A Modal Logic for Simulation, Refinement and Mutual Ignorance

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What is a simulation?

Given the requirements **atoms**, **forth**, **back** of bisimulation, a simulation relation satisfies **atoms** and **forth**, and a refinement relation satisfies **atoms** and **back**. Given a model (M, w), a model (M', w') is a simulation (refinement) of (M, w), if there is a simulation relation (refinement relation) between them.

Given a singleton S5 model where p is true, the two-state S5 model consisting of a p and a $\neg p$ state, indistinguishable by the agent, is a simulation. In the former the agent knows p, in the latter not.

Refinements encode information gain. It is belief expansion. Refinement modal logic is from 2014 (Bozzelli et al.). Simulations encode information loss. It is belief contraction. Simulation modal logic is from 2019 (Xing et al.) We advance Huili Xing's work.

We also introduce origin modal logic, to reason about mutual factual ignorance: all agents consider all valuations possible.

Syntax, structures, semantics

Language: $\mathcal{L} \ni \varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box_a\varphi \mid [\leftleftarrows]\varphi \mid [\rightrightarrows]\varphi \mid [\circ]\varphi$
— $[\leftleftarrows]\varphi$: after any refinement, φ (is true)
— $[\rightrightarrows]\varphi$: after any simulation, φ (is true)
— $[\circ]\varphi$: originally φ (was true)
— cover modality $\nabla_a\Phi$: $\bigwedge_{\varphi\in\Phi}\Diamond_a\varphi \wedge \Box_a \bigvee_{\varphi\in\Phi}\varphi$

Structures: *(Epistemic) Model* $M = (S, R, V)$ has *domain* S of *states*, *accessibility relations* $R_a \subseteq S^2$, *valuation* $V : S \to \mathcal{P}(P)$.
Mutual factual ignorance model M° has domain consisting of all valuations, with the universal relation between them for all agents.

A *bisimulation* $Z \subseteq S \times S'$ between $M = (S, R, V)$ and $M' = (S', R', V')$, notation $Z : M \leftrightarrows M'$ (existence: $M \leftrightarrows M'$):
— **atoms**: $V(s) = V'(s')$
— **forth**: if $R_a st$, then there is a $t' \in S'$ such that $R'_a s' t'$ and Ztt'
— **back**: if $R'_a s' t'$, then there is a $t \in S$ such that $R_a st$ and Ztt'
Similarly, a *simulation* $M \rightrightarrows M'$ satisfies **atoms** and **forth**, and a refinement $M \leftleftarrows M'$, satisfies **atoms** and **back**.
We also need *Q-restricted bisimulation* and *n-bounded bisimulation*.

Syntax, structures, semantics

Assume an epistemic model $M = (S, R, V)$, and let $s \in S$. We define $M_s \models \varphi$ (for: M_s *satisfies* φ, or φ is *true* in M_s) by induction.

$$
\begin{array}{lll}
M_s \models p & \text{iff} & p \in V(s) \\
M_s \models \neg\varphi & \text{iff} & M_s \not\models \varphi \\
M_s \models \varphi \wedge \psi & \text{iff} & M_s \models \varphi \text{ and } M_s \models \psi \\
M_s \models \Box_a \varphi & \text{iff} & M_t \models \varphi \text{ for all } t \in S \text{ such that } (s,t) \in R_a \\
M_s \models [\leftleftarrows]\varphi & \text{iff} & M'_{s'} \models \varphi \text{ for all } M'_{s'} \text{ such that } M_s \leftleftarrows M'_{s'} \\
M_s \models [\rightrightarrows]\varphi & \text{iff} & M'_{s'} \models \varphi \text{ for all } M'_{s'} \text{ such that } M_s \rightrightarrows M'_{s'} \\
M_s \models [\circ]\varphi & \text{iff} & M^{\circ}_{V(s)} \models \varphi
\end{array}
$$

Examples of refinement and simulation

Arbitrary models:

$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$	M	
$\bullet \longrightarrow \bullet \longrightarrow \bullet$	M'	submodel of M
$\boldsymbol{\bullet \longleftarrow \bullet \longrightarrow \bullet \longrightarrow \bullet}$	$\boldsymbol{M''}$	refinement of M, submodel of M'''
$\bullet \longleftarrow \bullet \longleftarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$	M'''	bisimilar copy of M

Epistemic models, two agents a, b

$$M: \quad \neg p \;\overset{a}{\text{———}}\; p \;\overset{b}{\text{———}}\; \overset{s}{p} \;\overset{a}{\text{———}}\; p \;\overset{b}{\text{———}}\; \neg p$$

$$M'': \quad \neg p \;\overset{ab}{\text{———}}\; p \;\overset{ab}{\text{———}}\; \overset{s}{p} \;\overset{ab}{\text{———}}\; p \;\overset{ab}{\text{———}}\; \neg p$$

$$M^\circ: \quad \overset{s}{p} \;\overset{ab}{\text{———}}\; \neg p$$

In M_s, a and b know that p is true, but are uncertain whether the other one knows. Mutual factual ignorance model M° is a simulation of M. It does not contain M, but M is a refinement of bisimilar copy M'' of M°. M° can be seen as a previous state of information wherein a and b were (partially) informed that p is true.

Properties of refinement, simulation, and origin

Refinement and simulation is about pruning and growing trees.
Both relations are *reflexive*, *transitive*, *confluent*, and *atomic*.
Atomic : the most informative message cuts the whole tree (removes all branches), the most forgetful message (about facts) brings you in the ignorance model.

$[\leftleftarrows]\varphi \rightarrow \varphi$	$(\mathbf{T}^{\leftleftarrows})$	$[\rightrightarrows]\varphi \rightarrow \varphi$	$(\mathbf{T}^{\rightrightarrows})$
$\langle\leftleftarrows\rangle\langle\leftleftarrows\rangle\varphi \rightarrow \langle\leftleftarrows\rangle\varphi$	$(\mathbf{4}^{\leftleftarrows})$	$\langle\rightrightarrows\rangle\langle\rightrightarrows\rangle\varphi \rightarrow \langle\rightrightarrows\rangle\varphi$	$(\mathbf{4}^{\rightrightarrows})$
$\langle\leftleftarrows\rangle[\leftleftarrows]\varphi \rightarrow [\leftleftarrows]\langle\leftleftarrows\rangle\varphi$	$(\mathbf{CR}^{\leftleftarrows})$	$\langle\rightrightarrows\rangle[\rightrightarrows]\varphi \rightarrow [\rightrightarrows]\langle\rightrightarrows\rangle\varphi$	$(\mathbf{CR}^{\rightrightarrows})$
$[\leftleftarrows]\langle\leftleftarrows\rangle\varphi \rightarrow \langle\leftleftarrows\rangle[\leftleftarrows]\varphi$	$(\mathbf{MK}^{\leftleftarrows})$	$[\rightrightarrows]\langle\rightrightarrows\rangle\varphi \rightarrow \langle\rightrightarrows\rangle[\rightrightarrows]\varphi$	$(\mathbf{MK}^{\rightrightarrows})$

Quite a bit is known about refinement modal logic:
— $\langle\leftleftarrows\rangle\varphi$ iff $\exists p\langle p\rangle\varphi$ bisimulation quantifier $\exists p$, announcement p
— $\langle\leftleftarrows\rangle\varphi$ iff $\langle\otimes\rangle\varphi$ quantifying over (finite) action models
— $\langle\leftleftarrows\rangle\varphi$ iff $\langle U\rangle\varphi$ synthesis of multipointed action model
Mutual factual ignorance model M° (given agents A and atoms P):
— every model is a refinement of M°

Axiomatizations - basic and origin modal logic

Axiomatization **ML**:

Prop	all substitution instances of tautologies of propositional logic
K	$\Box_a(\varphi \to \psi) \to (\Box_a\varphi \to \Box_a\psi)$
MP	from $\varphi \to \psi$ and φ infer ψ
N	from φ infer $\Box_a\varphi$
RE	from $\chi \leftrightarrow \psi$ infer $\varphi[\chi/p] \leftrightarrow \varphi[\psi/p]$

Axiomatization **OML** = **ML** + **ORI** where **ORI** is:

O1	$[\circ]\varphi_0 \leftrightarrow \varphi_0$	where $\varphi_0 \in \mathcal{L}_0$ (booleans)
OT	$[\circ](\Box_a\varphi \to \varphi)$	
O5	$[\circ](\Diamond_a\varphi \to \Box_a\Diamond_a\varphi)$	
OExch	$[\circ](\Box_a\varphi \to \Box_b\varphi)$	
OFull	$[\circ]\Diamond_a\varphi$	φ of form $\bigwedge_{p\in Q_1} p \wedge \bigwedge_{p \in Q_2} \neg p$ with $Q_1 \cap Q_2 = \emptyset$
ODual	$[\circ]\neg\varphi \leftrightarrow \neg[\circ]\varphi$	
ODisj	$[\circ](\varphi \vee \psi) \leftrightarrow ([\circ]\varphi \vee [\circ]\psi)$	
OMP	from $[\circ](\varphi \to \psi)$ and $[\circ]\varphi$ infer $[\circ]\psi$	
ON	from $[\circ]\varphi$ infer $[\circ]\Box_a\varphi$	

Axiomatizations — refinement and simulation modal logic

Axiomatization **RML** = **ML** + **REF** where **REF** is:

RQ1	$\langle\Leftarrow\rangle\varphi_0 \leftrightarrow \varphi_0$	where $\varphi_0 \in \mathcal{L}_0$
RQ2	$\langle\Leftarrow\rangle(\varphi \vee \psi) \leftrightarrow (\langle\Leftarrow\rangle\varphi \vee \langle\Leftarrow\rangle\psi)$	
RQ3	$\langle\Leftarrow\rangle(\varphi_0 \wedge \varphi) \leftrightarrow (\varphi_0 \wedge \langle\Leftarrow\rangle\varphi)$	where $\varphi_0 \in \mathcal{L}_0$
RQ4	$\langle\Leftarrow\rangle \bigwedge_{a\in A} \nabla_a \Phi_a \leftrightarrow \bigwedge_{a\in A} \bigwedge_{\varphi\in\Phi_a} \Diamond_a \langle\Leftarrow\rangle\varphi$	

Axiomatizion **SML** = **ML** + **SIM**$_{\text{cons}}$ where **SIM**$_{\text{cons}}$ is:

SQ1	$\langle\Rightarrow\rangle\varphi_0 \leftrightarrow \varphi_0$	where $\varphi_0 \in \mathcal{L}_0$
SQ2	$\langle\Rightarrow\rangle(\varphi \vee \psi) \leftrightarrow (\langle\Rightarrow\rangle\varphi \vee \langle\Rightarrow\rangle\psi)$	
SQ3	$\langle\Rightarrow\rangle(\varphi_0 \wedge \varphi) \leftrightarrow (\varphi_0 \wedge \langle\Rightarrow\rangle\varphi)$	where $\varphi_0 \in \mathcal{L}_0$
SQ4$_{\text{cons}}$	$\langle\Rightarrow\rangle \bigwedge_{a\in A} \nabla_a \Phi_a \leftrightarrow \bigwedge_{a\in A} \Box_a \bigvee_{\varphi\in\Phi_a} \langle\Rightarrow\rangle\varphi$	all φ consist.

Axiomatization **ROSML** = **ML** + **REF** + **ORI** + **SIM** where we get **SIM** from **SIM**$_{\text{cons}}$ when replacing **SQ4**$_{\text{cons}}$ by **SQ4**:

SQ4 $\langle\Rightarrow\rangle \bigwedge_{a\in A} \nabla_a \Phi_a \leftrightarrow \bigwedge_{a\in A} (\Box_a \bigvee_{\varphi\in\Phi_a} \langle\Rightarrow\rangle\varphi \wedge [\circ] \bigwedge_{\varphi\in\Phi_a} \Diamond_a \langle\Leftarrow\rangle\varphi)$

Soundness and completeness

Soundness of all axiomatizations is pretty standard, except for the axioms rewriting refinement or simulation binding cover modalities.

RQ4	$\langle\leftleftarrows\rangle \bigwedge_{a\in A} \nabla_a \Phi_a \leftrightarrow \bigwedge_{a\in A} \bigwedge_{\varphi\in\Phi_a} \Diamond_a \langle\leftleftarrows\rangle\varphi$	
SQ4_{cons}	$\langle\rightrightarrows\rangle \bigwedge_{a\in A} \nabla_a \Phi_a \leftrightarrow \bigwedge_{a\in A} \Box_a \bigvee_{\varphi\in\Phi_a} \langle\rightrightarrows\rangle\varphi$	all φ consist.
SQ4	$\langle\rightrightarrows\rangle \bigwedge_{a\in A} \nabla_a \Phi_a \leftrightarrow \bigwedge_{a\in A}(\Box_a \bigvee_{\varphi\in\Phi_a} \langle\rightrightarrows\rangle\varphi \wedge [\circ] \bigwedge_{\varphi\in\Phi_a} \Diamond_a \langle\leftleftarrows\rangle\varphi)$	

— refinement covers a set of formulas, iff all formulas were possible
— simulation covers a set of formulas, iff some formula was necessary
— consistency is explicit (SQ4_{cons}) or enforced (SQ4). Counterexample:

$$\langle\rightrightarrows\rangle(\Diamond\top \wedge \Diamond\bot) \leftrightarrow \Box(\langle\rightrightarrows\rangle\top \vee \langle\rightrightarrows\rangle\bot) \qquad \text{is invalid!}$$

Completeness of all these systems is pretty standard: we 'push' the $\leftleftarrows$, $\rightrightarrows$ and $\circ$ modalities 'inward' by rewriting, until they bind a boolean, in which case they disappear. For **OML** somewhat complex normal forms are required for that rewriting. All logics are as expressive as basic modal logic **ML**. But far more complex, e.g. the complexity of **RML** satisfiability is probably non-elementary.

Relation to other approaches

- Xing et al. Covariant-contravariant RML, FLAP 2019.
  A CC-refinement is a (B, C)-refinement relation where B, C are disjoint subsets of A. Their $(\emptyset, A)$-refinement we call a refinement and their $(A, \emptyset)$-refinement we call a simulation. Our axiom $\mathbf{SQ4}_{\text{cons}}$ is theirs, modulo a lot of syntactic sugar.
- Balbiani et al. *Before Announcement*, AiML 2016
  Baltag et al. *APAL with Memory*, JPL 2022
  These approaches quantify over announcements and backwards. There, the initial epistemic model is explicit and arbitrary: there, $M = (S^{\text{initial}}, S, \sim, V)$; here $M = (S, \sim, V)$ inducing $M^\circ = (\mathcal{P}(P), \sim^\circ, V^\circ)$.
- Areces et al. *Relation-changing modal operators*, IGPL 2015
  Relation to *bridging modalities*, encoding belief contraction.
- vD et al. *Arrow Update Synthesis*. Inform. & Comput. 2020.
  Similar axiomatization to **RML**. Refinement is (nearly) update equivalent to quantifying over arrow update models.

Further research

- simulation epistemic logic
  Quantifies over models with equivalence relations (knowledge). Axiomatization does not extend **SML**: axiom **SQ4** is now invalid.
- complexity of model checking and satisfiability
  Unknown if results for single/multi-agent RML transfer to SML.
- belief contraction in dynamic epistemic logic
  Public announcement logic and action model logic do belief expansion; belief revision in DEL uses (enriched) plausibility models. What is a mechanism for belief contraction in DEL?
- temporal epistemic embeddings
  There is no relation between time and action in DEL. SML is about information loss. This can be interpreted as reasoning over the past (or future). There are various, also asynchronous, embeddings of DEL into temporal epistemic logics.