Satisfiability of APAL with Common Knowledge is Σ_1^1-hard

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Plan of the Talk

Part I. Arbitrary Public Announcement Logic with Common Knowledge (APALC)

Part II. APALC and a Reduction from the Recurring Tiling Problem

Part III. Corollaries and Conclusion

Part I

APAL with Common Knowledge

Epistemic Logic

Agents and propositions

Let A and P be countable sets of agents and propositional variables

Language of EL

$$\mathscr{ELC} \ni \varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box_a \varphi$$

Epistemic models

An epistemic model M is a tuple $(S, \sim, V)$, where

- $S \neq \emptyset$ is a set of states;
- $\sim: A \to 2^{S\times S}$ is an indistinguishability function with each $\sim_a$ being an equivalence relation;
- $V : P \to 2^S$ is the valuation function.

Semantics of EL

$M_s \vDash p$ iff $s \in V(p)$

$M_s \vDash \neg\varphi$ iff $M_s \nvDash \varphi$

$M_s \vDash \varphi \land \psi$ iff $M_s \vDash \varphi$ and $M_s \vDash \psi$

$M_s \vDash \Box_a \varphi$ iff $\forall t \in S : s \sim_a t$ implies $M_t \vDash \varphi$

$M_s \vDash \Diamond_a \varphi$ iff $\exists t \in S : s \sim_a t$ and $M_t \vDash \varphi$

Theorem. EL has a sound and complete axiomatisation

Halpern, Moses. *A guide to completeness and complexity for modal logics of knowledge and belief*, 1992.

Public Announcement Logic

Language of PAL

$$\mathscr{PAL} \ni \varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box_a \varphi \mid [\varphi]\varphi$$

Semantics

$M_s \vDash [\psi]\varphi$ iff $M_s \vDash \psi$ implies $M_s^{\psi} \vDash \varphi$

$M_s \vDash \langle\psi\rangle\varphi$ iff $M_s \vDash \psi$ and $M_s^{\psi} \vDash \varphi$

Updated model

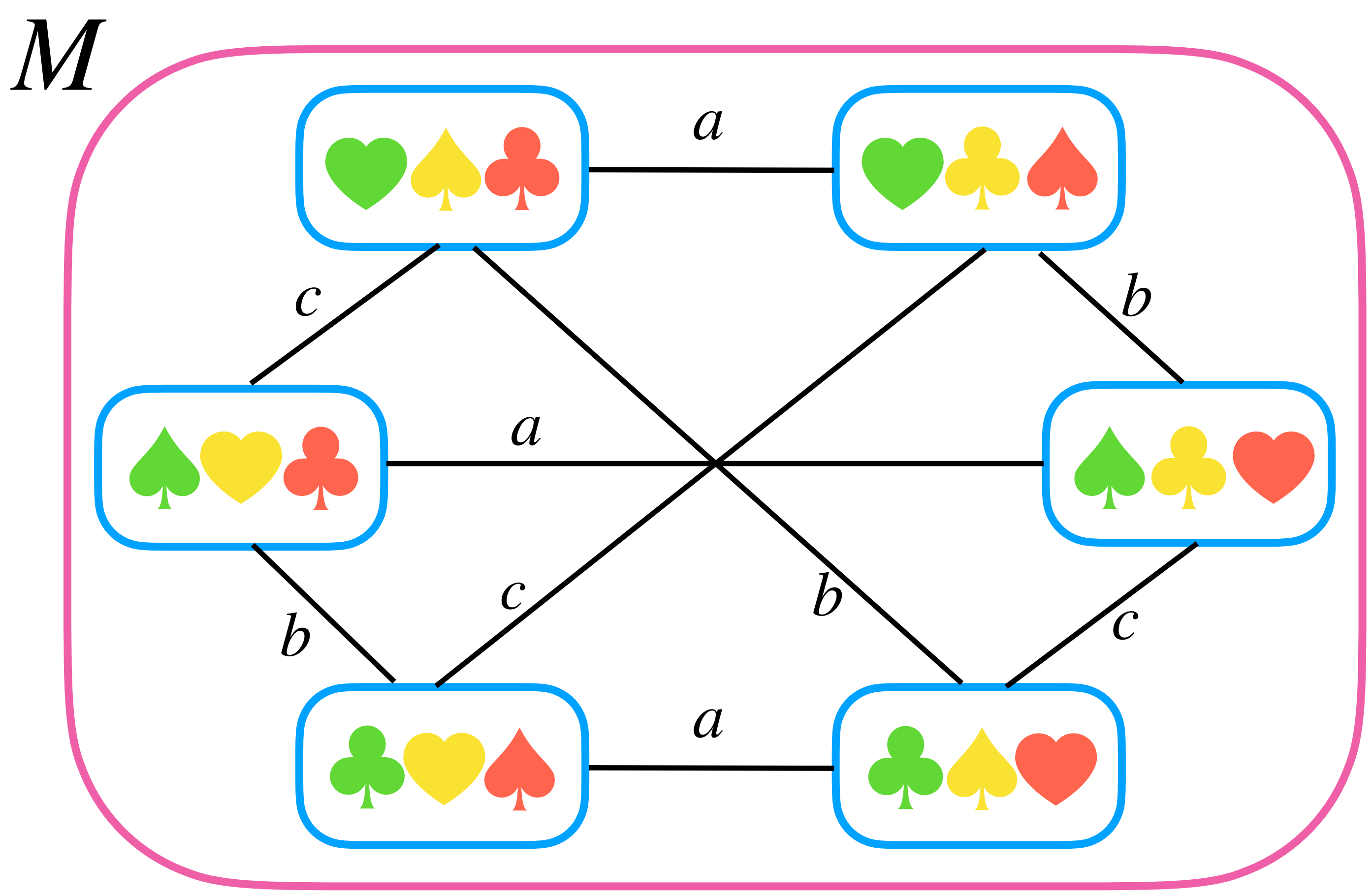
Let $M = (S, \sim, V)$ and $\varphi \in \mathscr{PAL}$. An updated model M^{φ} is a tuple $(S^{\varphi}, \sim^{\varphi}, V^{\varphi})$, where

- $S^{\varphi} = \{s \in S \mid M_s \vDash \varphi\}$;
- $\sim_a^{\varphi} = \sim_a \cap (S^{\varphi} \times S^{\varphi})$;
- $V^{\varphi}(p) = V(p) \cap S^{\varphi}$.

Van Ditmarsch, Van der Hoek, Kooi. *Dynamic Epistemic Logic*, Section 4. 2008.

Card Example

Three agents, Alice, Bob, and Carol, have each drawn one card from a deck of {♥ ♠ ♣}, and then Alice says that she does not have clubs



Alice says that she does not have clubs: $\neg ♣_a$

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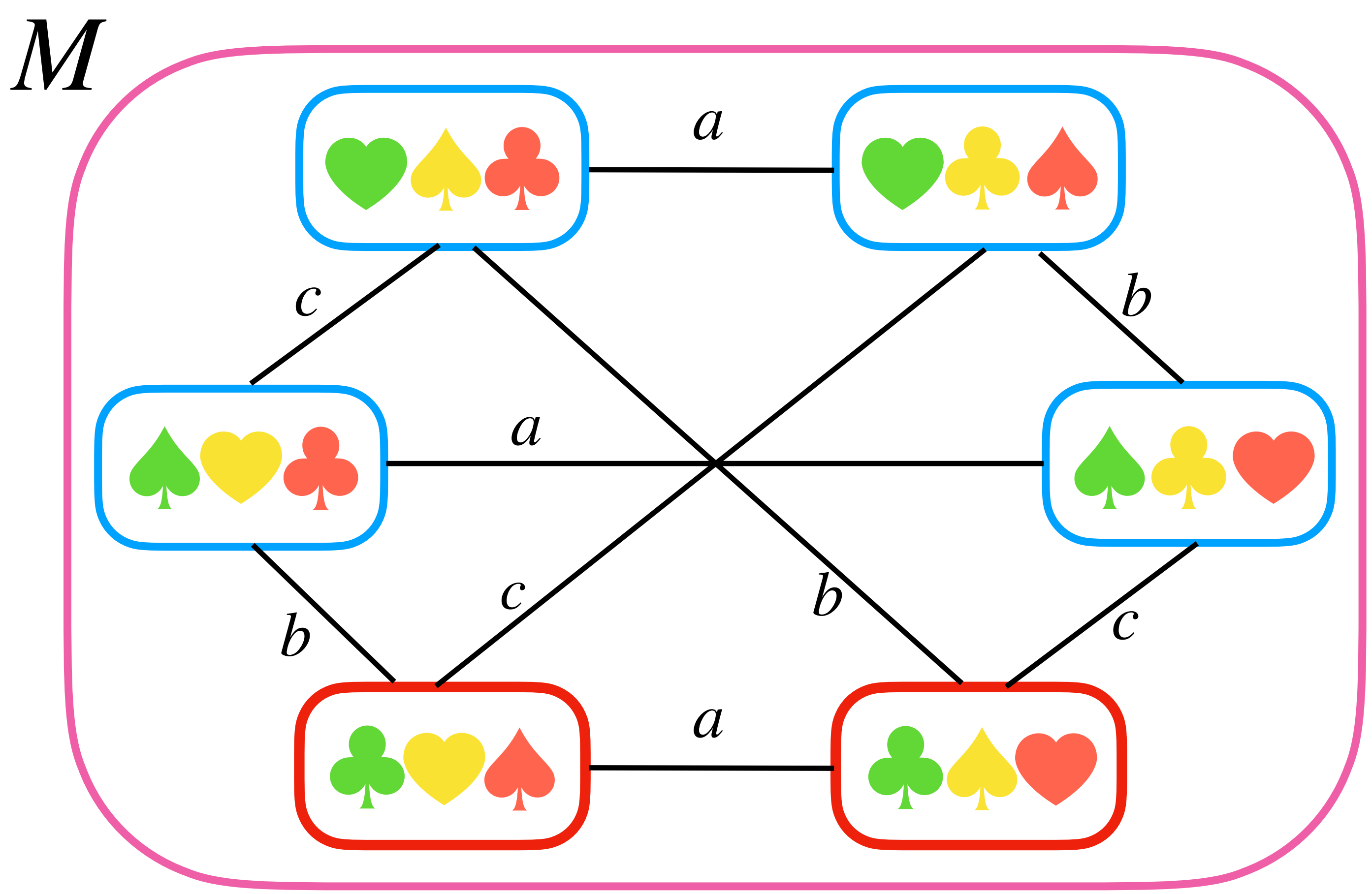

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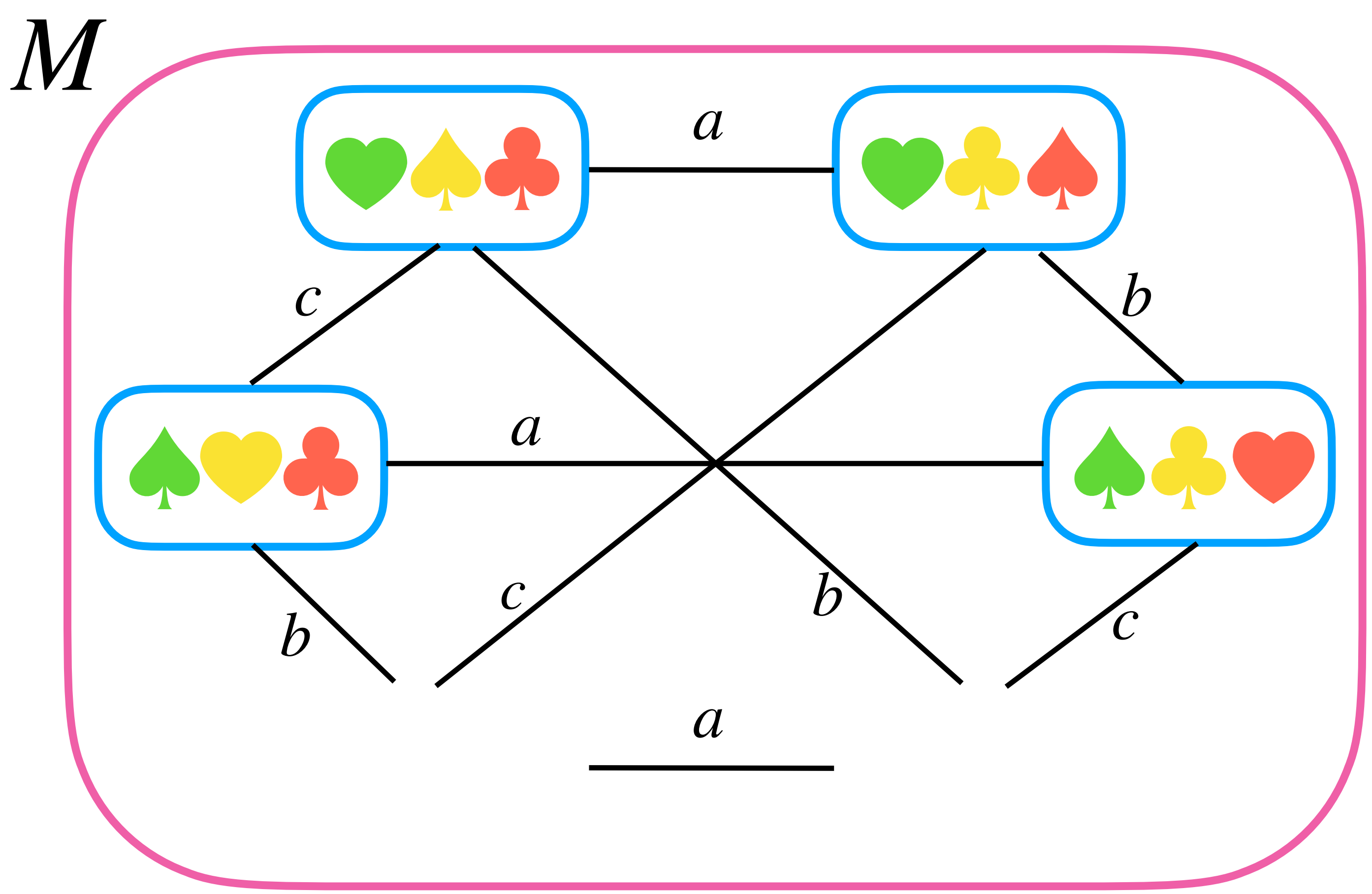

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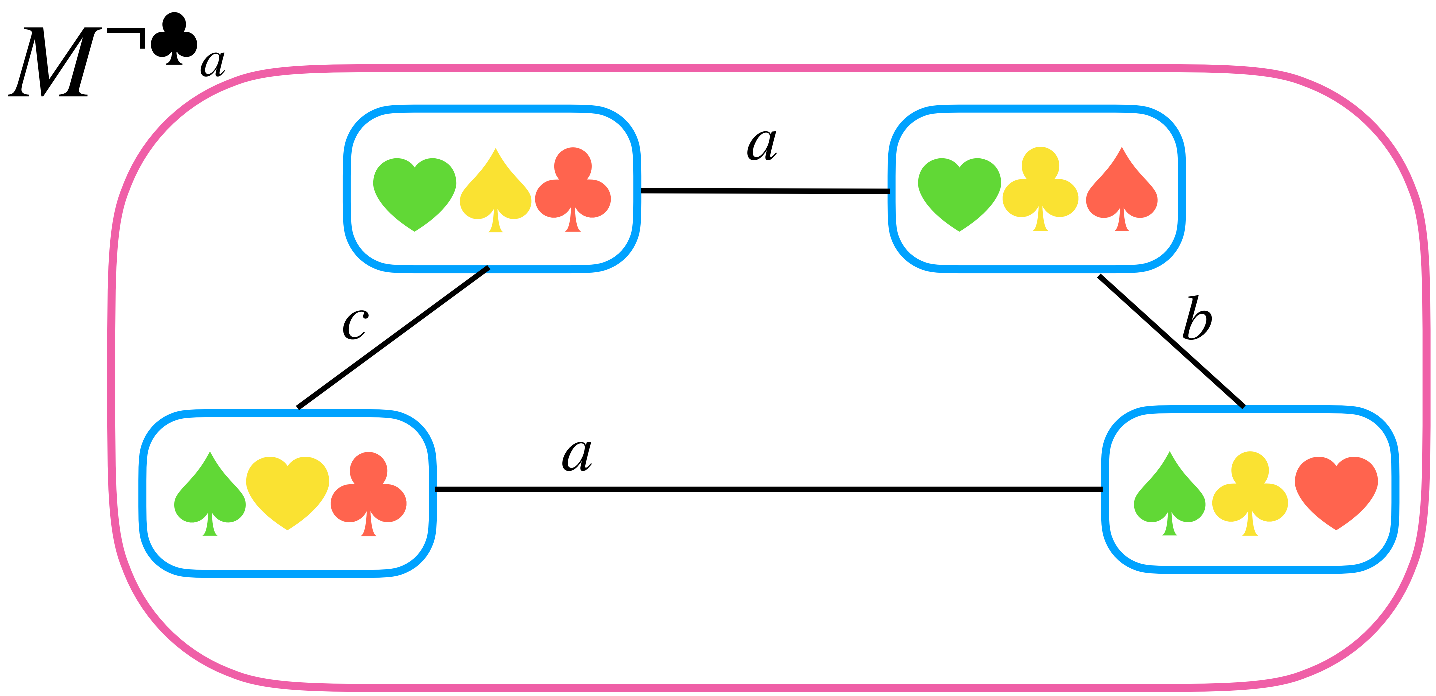

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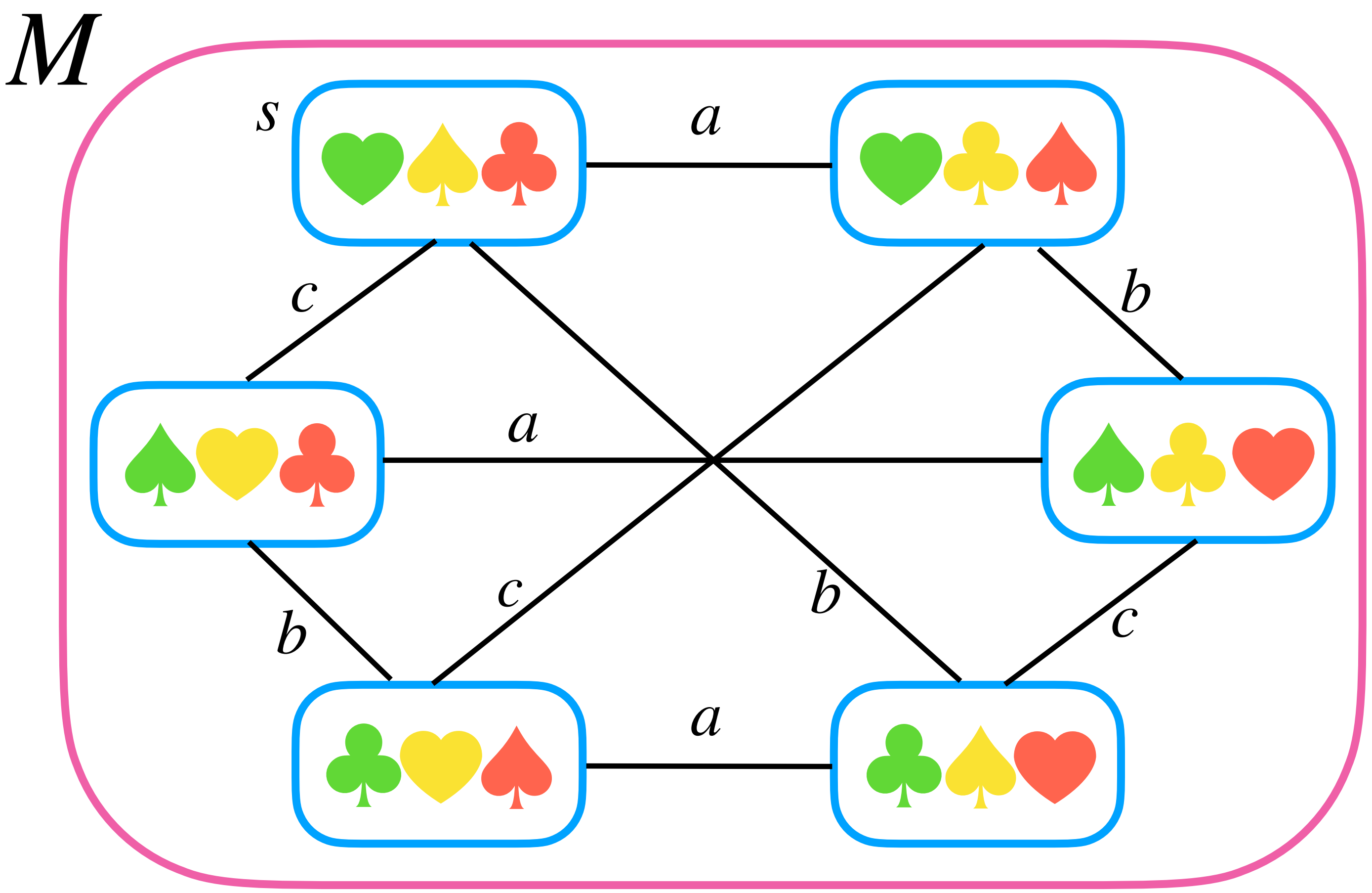


$M_s \models [\neg \clubsuit_a] \Box_b (\heartsuit_a \wedge \spadesuit_b \wedge \clubsuit_c)$

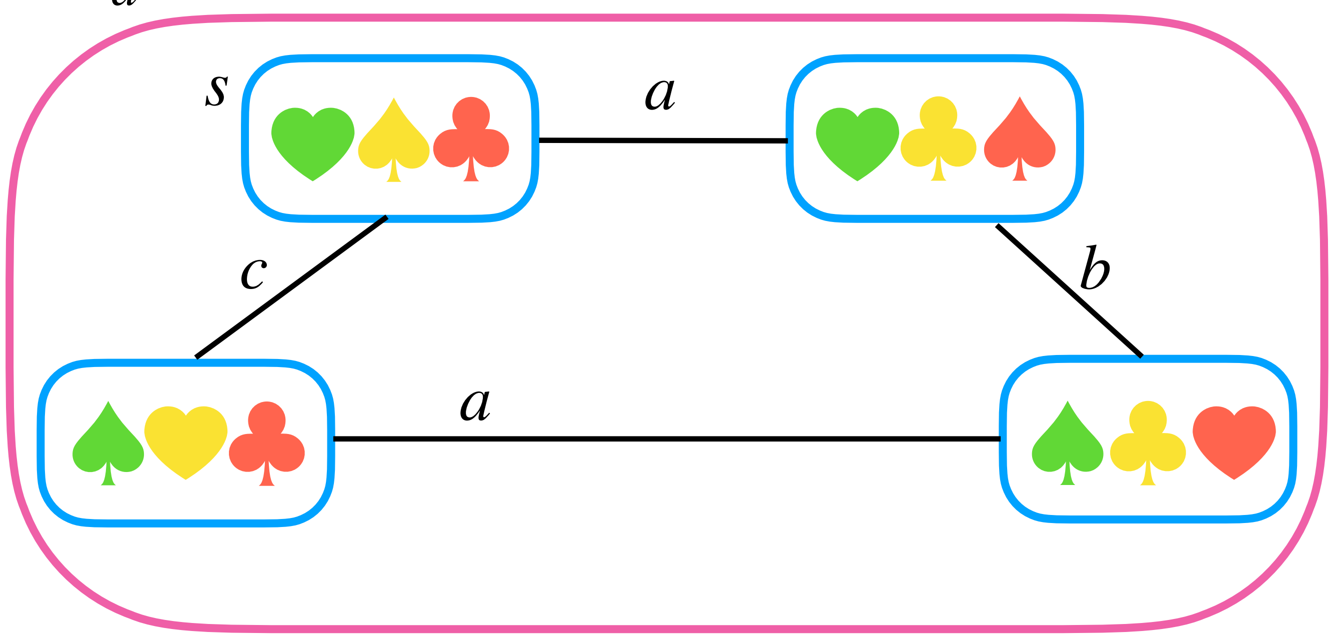
$[\psi]\varphi$: after public announcement of ψ, φ is true

Van Ditmarsch, Van der Hoek, Kooi. *Dynamic Epistemic Logic*, Section 4. 2008.

Card Example

Three agents, Alice, Bob, and Carol, have each drawn one card from a deck of {♥ ♠ ♣}, and then Alice says that she does not have clubs

$M^{\neg \clubsuit_a}$



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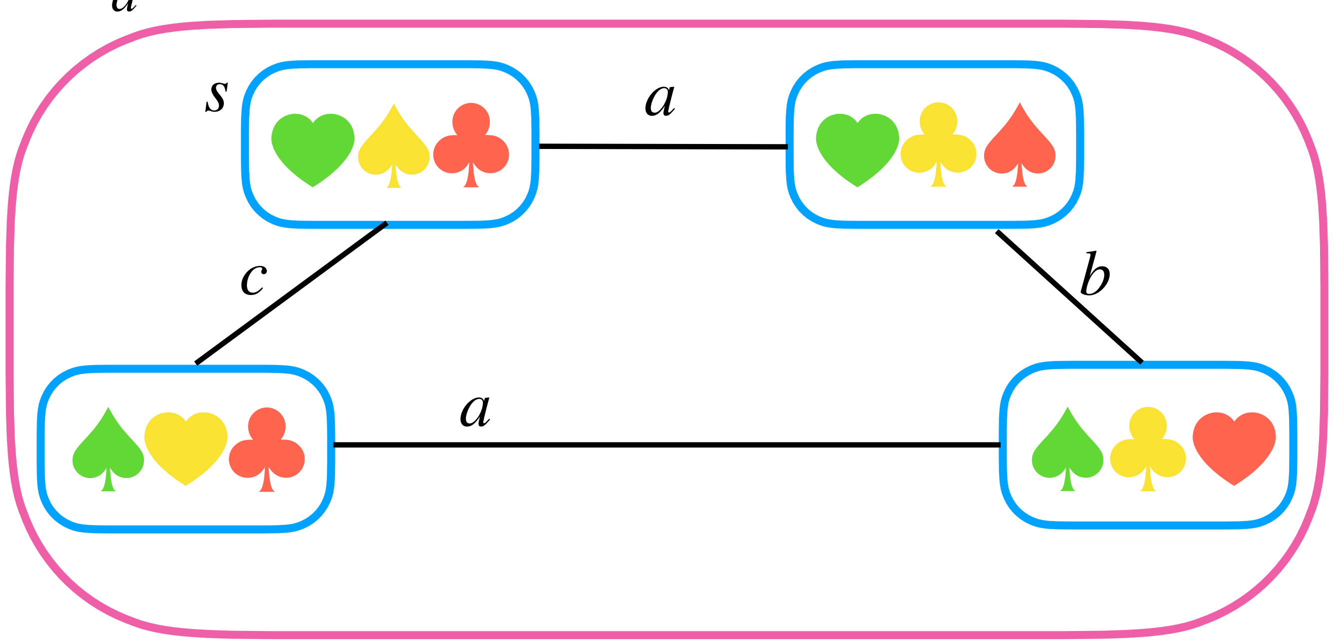


$M_s \models [\neg \clubsuit_a] \Box_b (\heartsuit_a \wedge \spadesuit_b \wedge \clubsuit_c)$

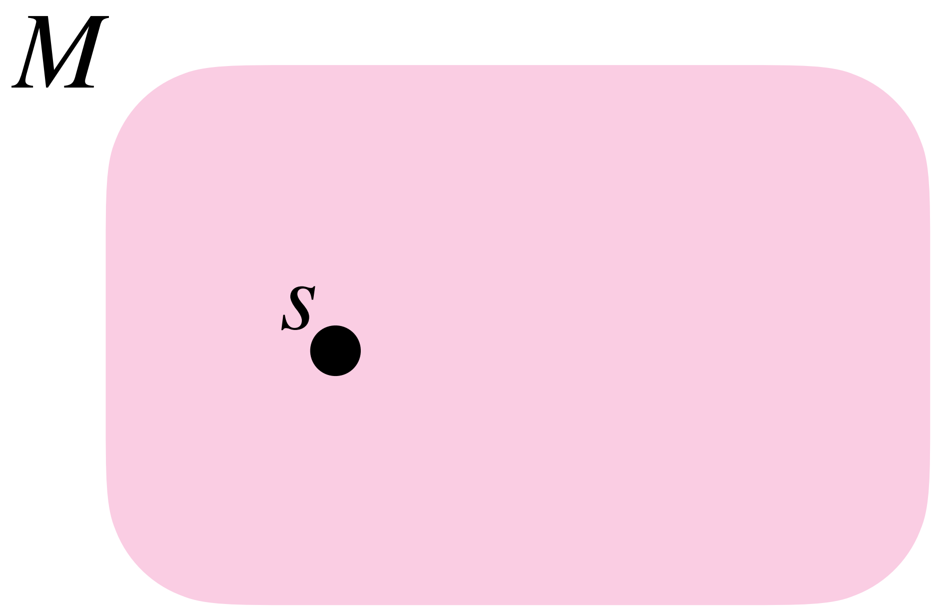
Theorem. PAL has a sound and complete axiomatisation

Theorem. PAL and EL are equally expressive

Axioms of PAL allow one to rewrite any formula of PAL into a formula of EL

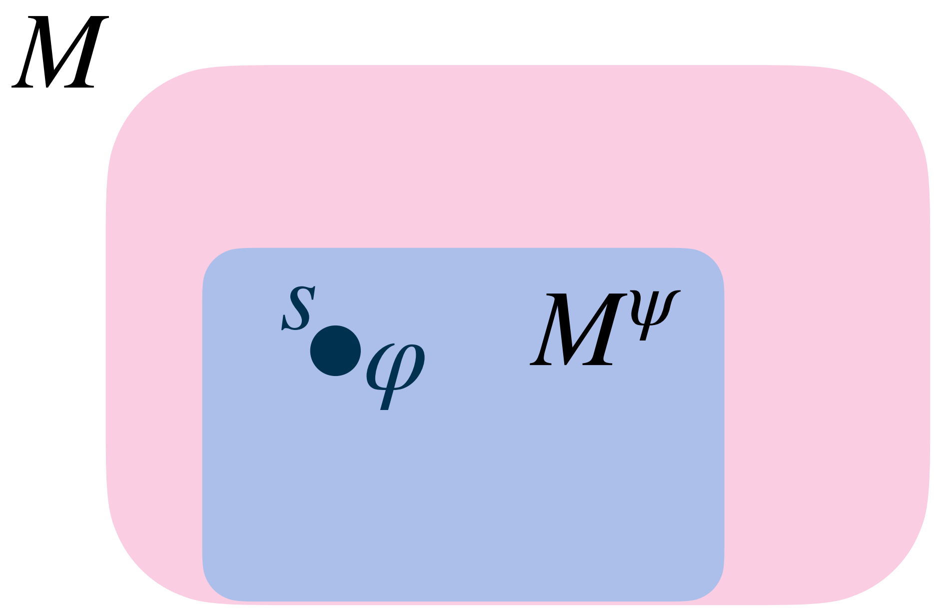
Van Ditmarsch, Van der Hoek, Kooi. *Dynamic Epistemic Logic*, Section 4. 2008.

Quantifying Over Public Announcements



$\langle ! \rangle \varphi$: There is a public announcement, after which φ is true

Quantifying Over Public Announcements



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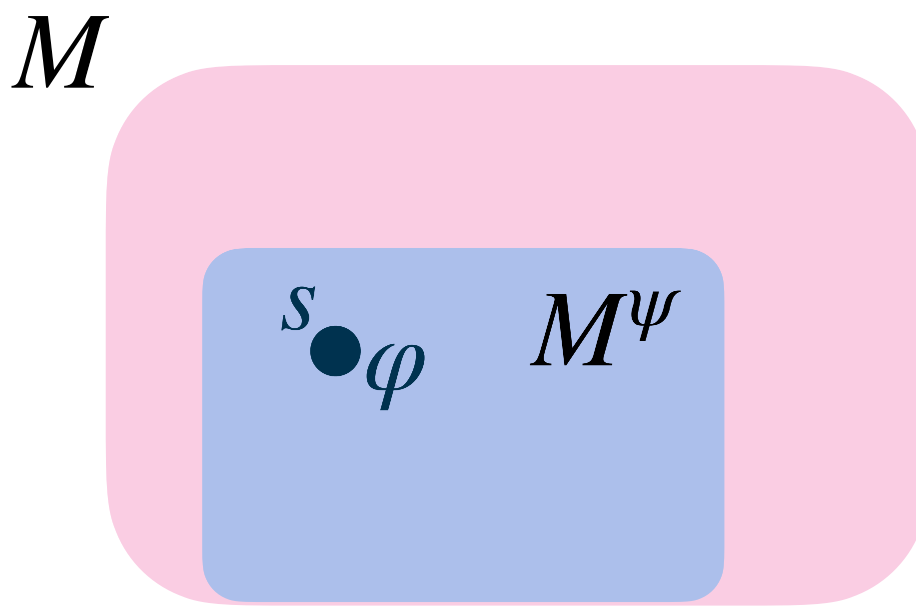
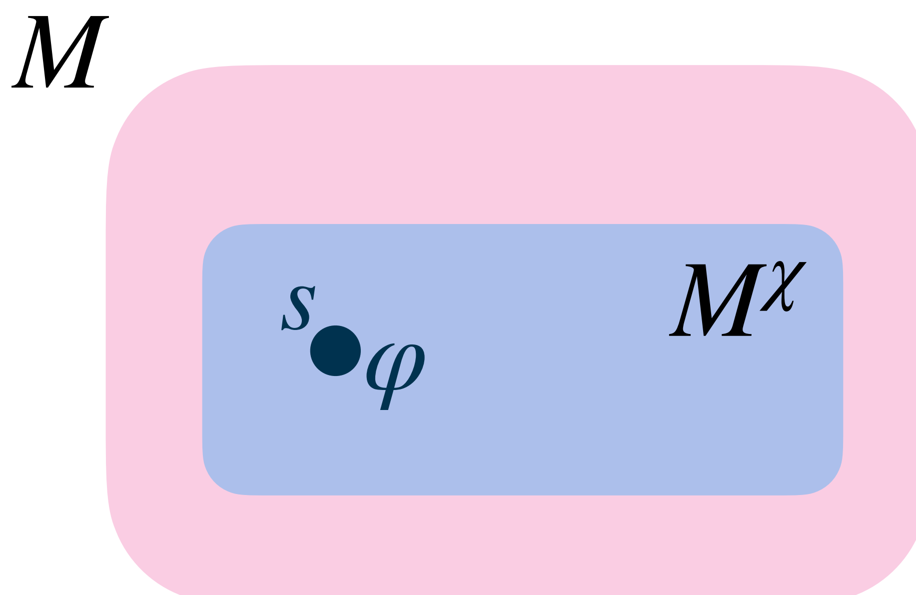


$[!]\varphi$: After all public announcements, φ is true

Quantifying Over Public Announcements



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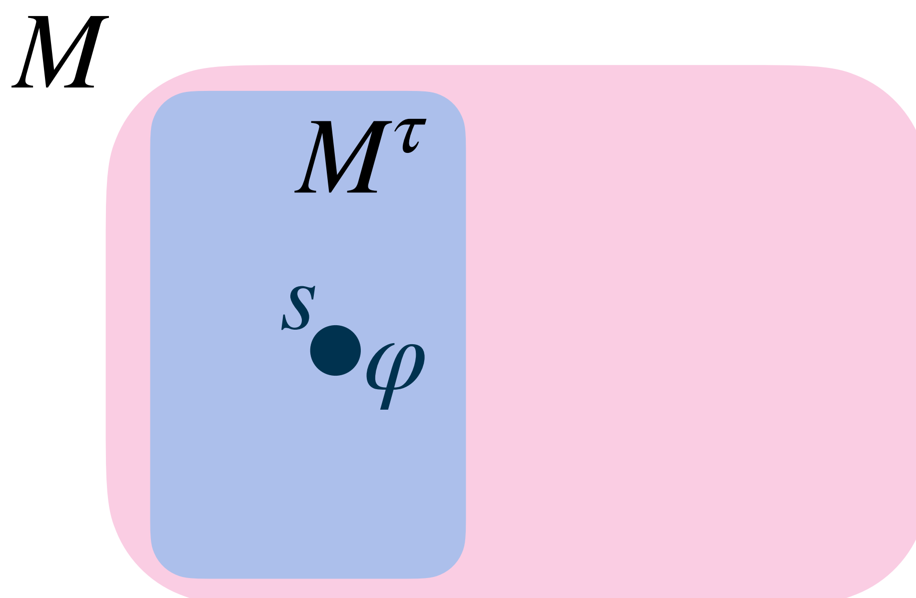


$[\,!\,]\varphi$: After all public announcements, φ is true

Arbitrary PAL

Language of APAL

$$\mathscr{APAL} \ni \varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box_a \varphi \mid [\varphi]\varphi \mid [!]\varphi$$

Semantics

$$M_s \vDash [!]\varphi \text{ iff } \forall\psi \in \mathscr{PAL} : M_s \vDash [\psi]\varphi$$

$$M_s \vDash \langle ! \rangle\varphi \text{ iff } \exists\psi \in \mathscr{PAL} : M_s \vDash \langle\psi\rangle\varphi$$

Some validities

$$\langle\psi\rangle\varphi \to \langle ! \rangle\varphi \qquad [!]\varphi \to \varphi$$

$$\langle ! \rangle\varphi \leftrightarrow \langle ! \rangle\langle ! \rangle\varphi \qquad \langle ! \rangle[!]\varphi \leftrightarrow [!]\langle ! \rangle\varphi$$

Quantification is restricted to formulas of PAL in order to avoid circularity

Balbiani et al. *'Knowable' as 'Known After an Announcement'*, 2008.

Axiomatisation of APAL

Axioms of EL and PAL

$[!]\varphi \to [\psi]\varphi$ with $\psi \in \mathscr{PAL}$

From $\{\eta([\psi]\varphi) \mid \psi \in \mathscr{PAL}\}$
infer $\eta([!]\varphi)$

Infinitary number of premises

Theorem. APAL is more expressive than PAL

Theorem. APAL is sound and complete

Open Problem. Is there a finitary axiomatisation of APAL?

Balbiani, Van Ditmarsch. *A simple proof of the completeness of APAL*, 2015.

Alternative Open Problem

Open Problem*. Is there a finitary axiomatisation of APAL with common knowledge?

Language of APALC $\mathscr{APALC} \ni \varphi ::= p \mid \neg\varphi \mid (\varphi \land \varphi) \mid \Box_a \varphi \mid \blacksquare_G \varphi \mid [\varphi]\varphi \mid [!]\varphi$

Semantics

$M_s \models \blacksquare_G \varphi$ iff $\forall t \in S : s \sim_G t$ and $M_t \models \varphi$

$M_s \models [!]\varphi$ iff $\forall \psi \in \mathscr{PALC} : M_s \models [\psi]\varphi$

$\blacksquare_G \varphi$: It is common knowledge among agents from group G that φ holds

$$\sim_G = (\bigcup_{a \in G} \sim_a)^*$$

Part II

APALC and the Reduction from the Recurring Tiling Problem

Recurring Tiling Problem

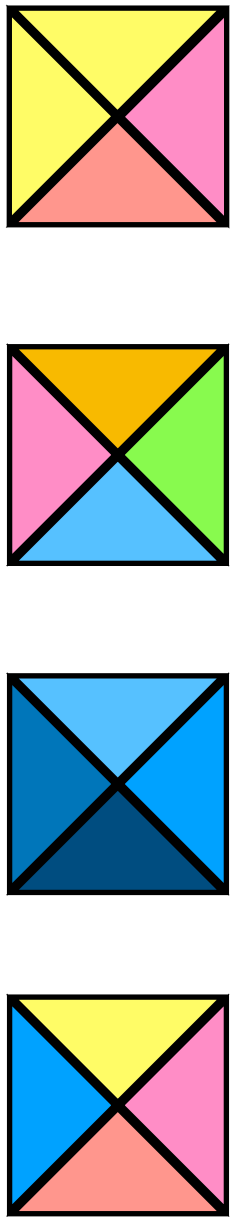
Given a finite set of colours C, a tile is a function

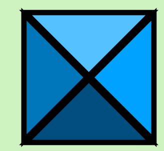
$$\tau : \{\mathtt{north}, \mathtt{south}, \mathtt{east}, \mathtt{west}\} \to C$$

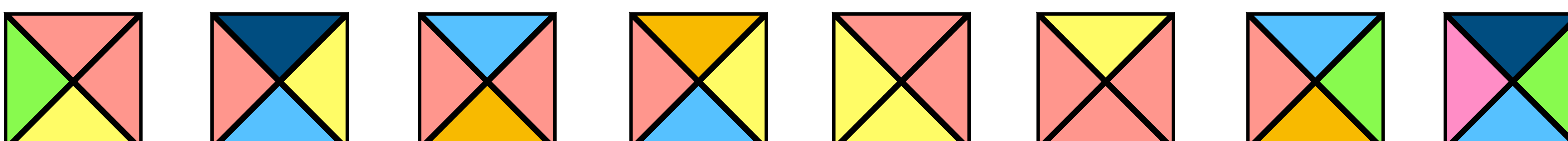
Given a finite set of tiles T, a tiling problem is the problem to determine whether T can tile the plane

Given a special tile τ^*, a recurring tiling problem is the problem to determine whether T can tile the plane such that τ^* appears infinitely often in the first column

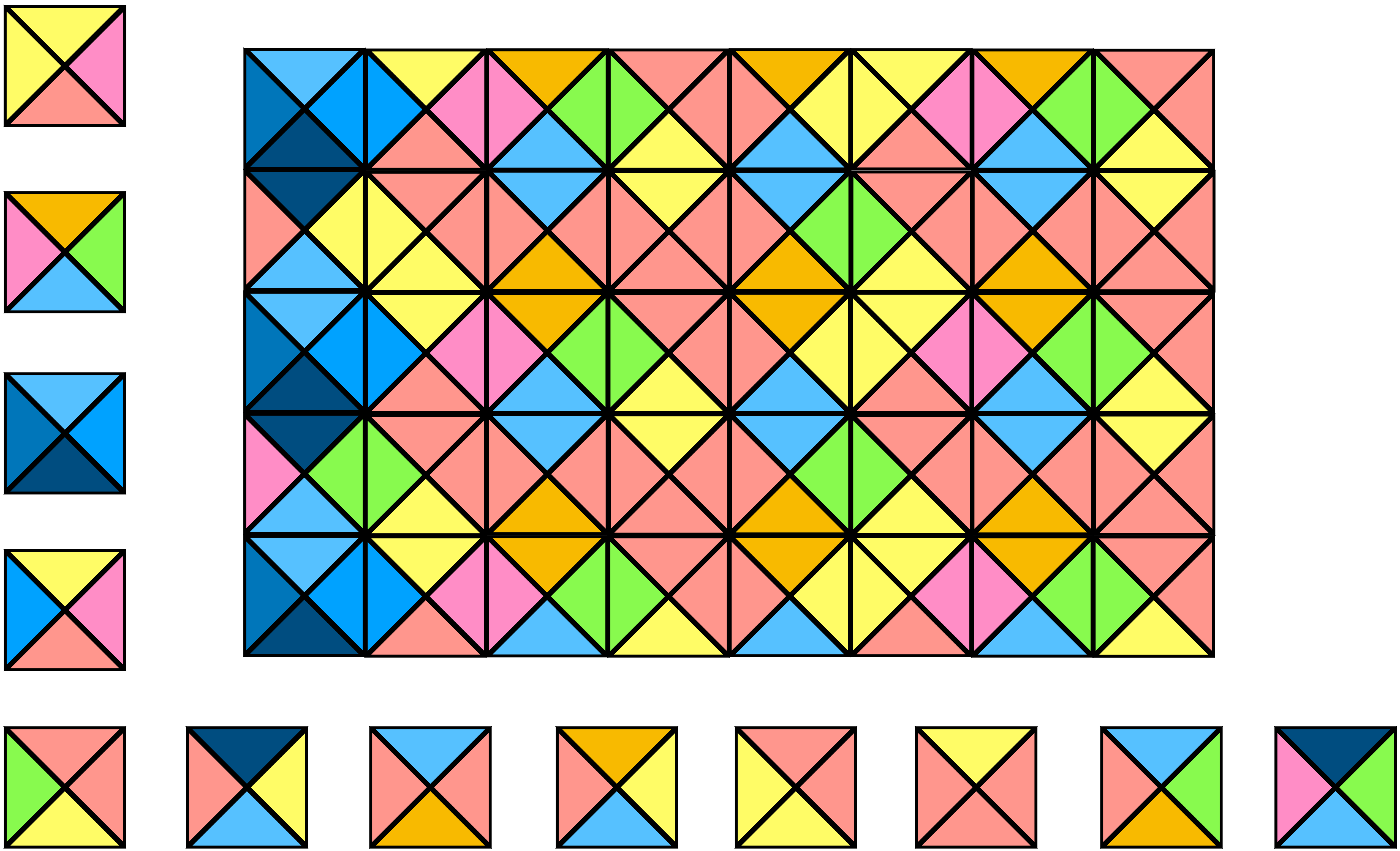
Recurring Tiling Problem



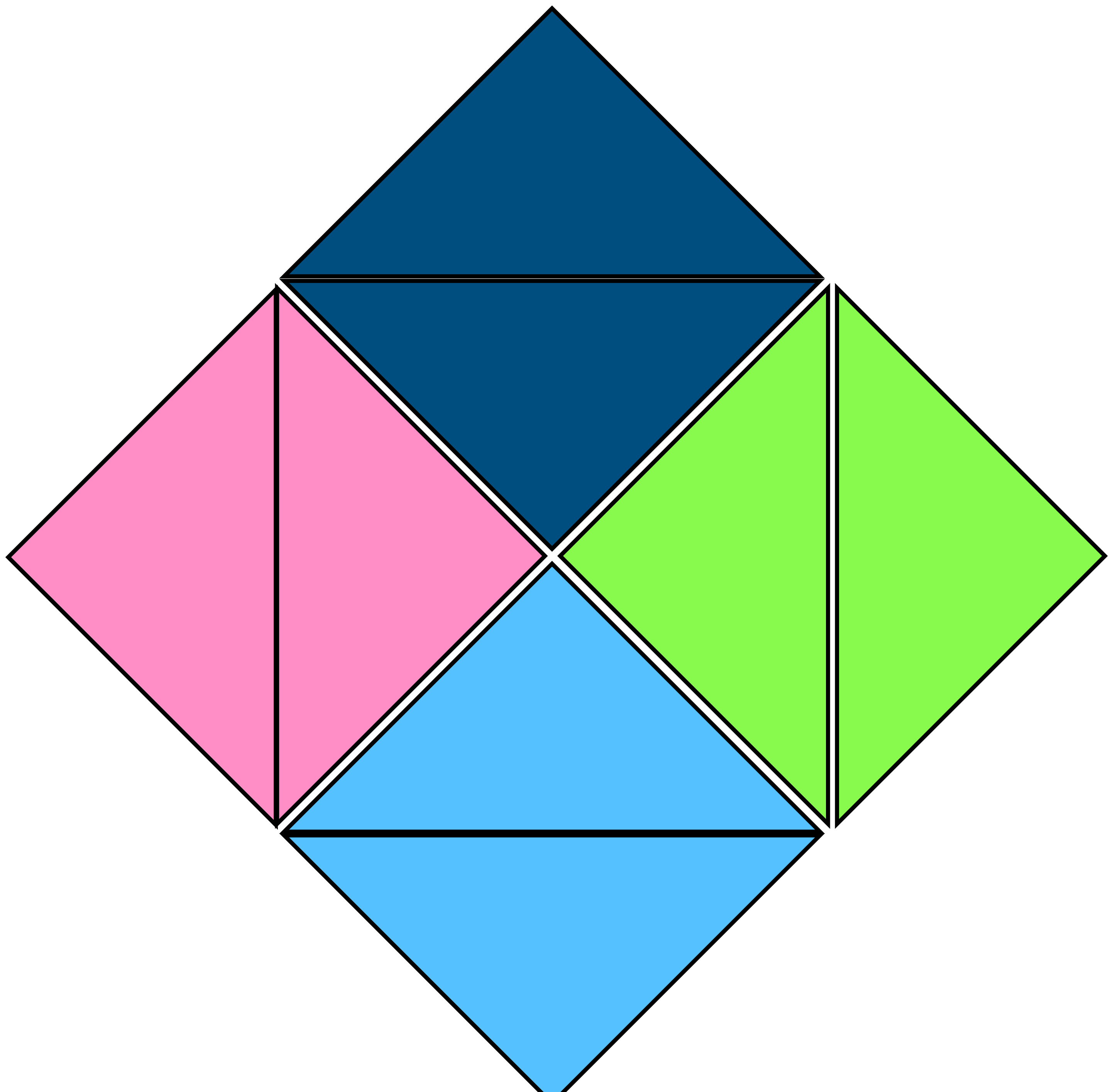
Can these tiles tile the plane such that  appears infinitely often in the first column?



Recurring Tiling Problem



Encoding a Tiling



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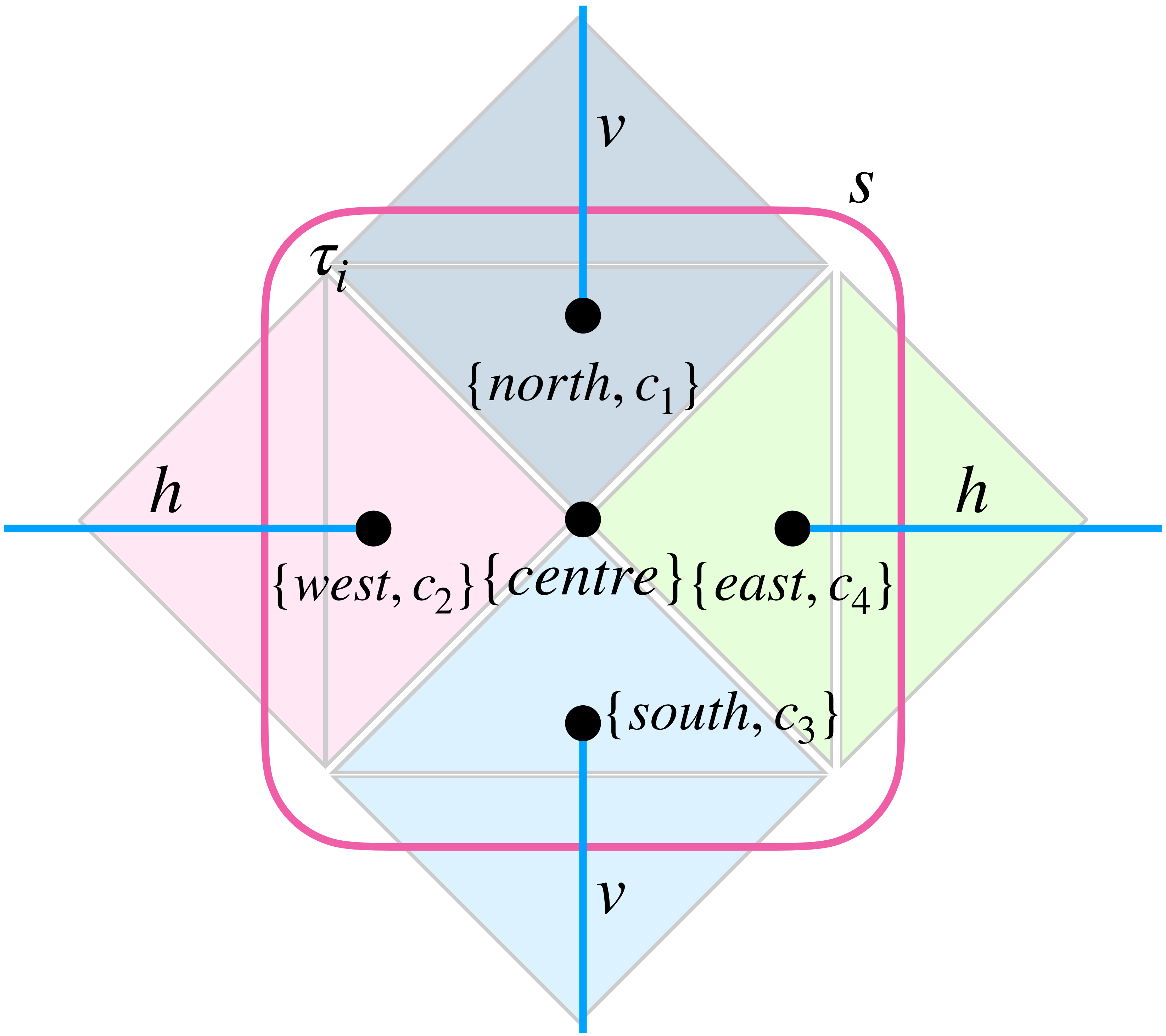

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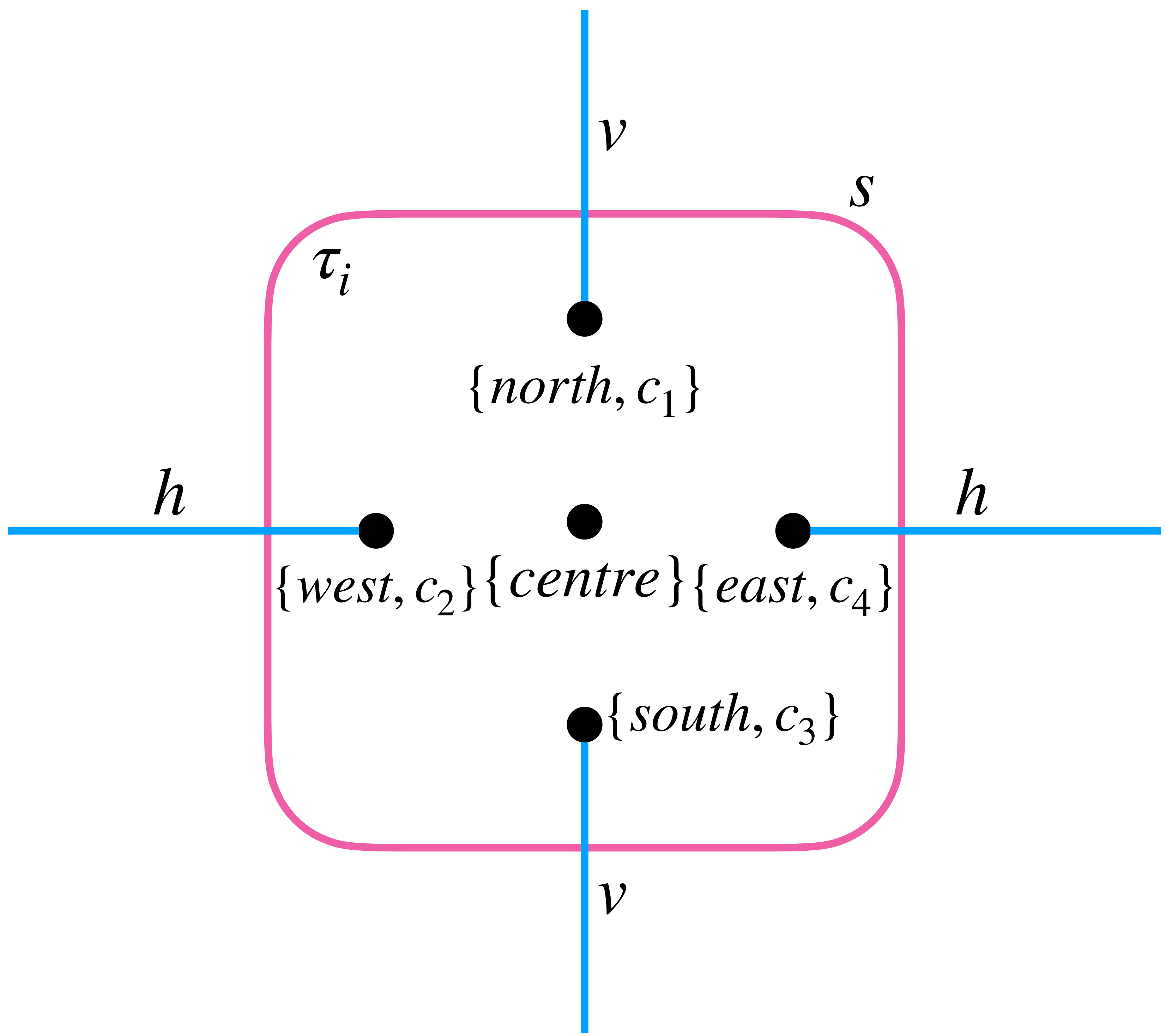

Encoding a Tiling

ψ_{tile} encodes the representation of a single tile

adj_tiles requires that adjoining tiles agree on colour

$init$ forces the existence of a tile at position $(0,0)$

$\psi_{x\&y}$ guarantees that making a move does not lead to different tiles

$tile\_left$ forces the special tile to appear only in the leftmost column

$$right \,\&\, up := [!](\Diamond_{right}\Diamond_{up}\mathtt{centre} \rightarrow \Box_{up}\Box_{right}\,\mathtt{centre})$$

Encoding a Tiling

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$$\Psi_T := \blacksquare_{\{h,v,s\}}(\psi_{tile} \wedge adj\_tiles \wedge init \wedge \psi_{x\&y} \wedge tile\_left)$$

Encoding a Tiling

$$\Psi_T := \blacksquare_{\{h,v,s\}}(\psi_{tile} \wedge adj\_tiles \wedge init \wedge \psi_{x\&y} \wedge tile\_left)$$

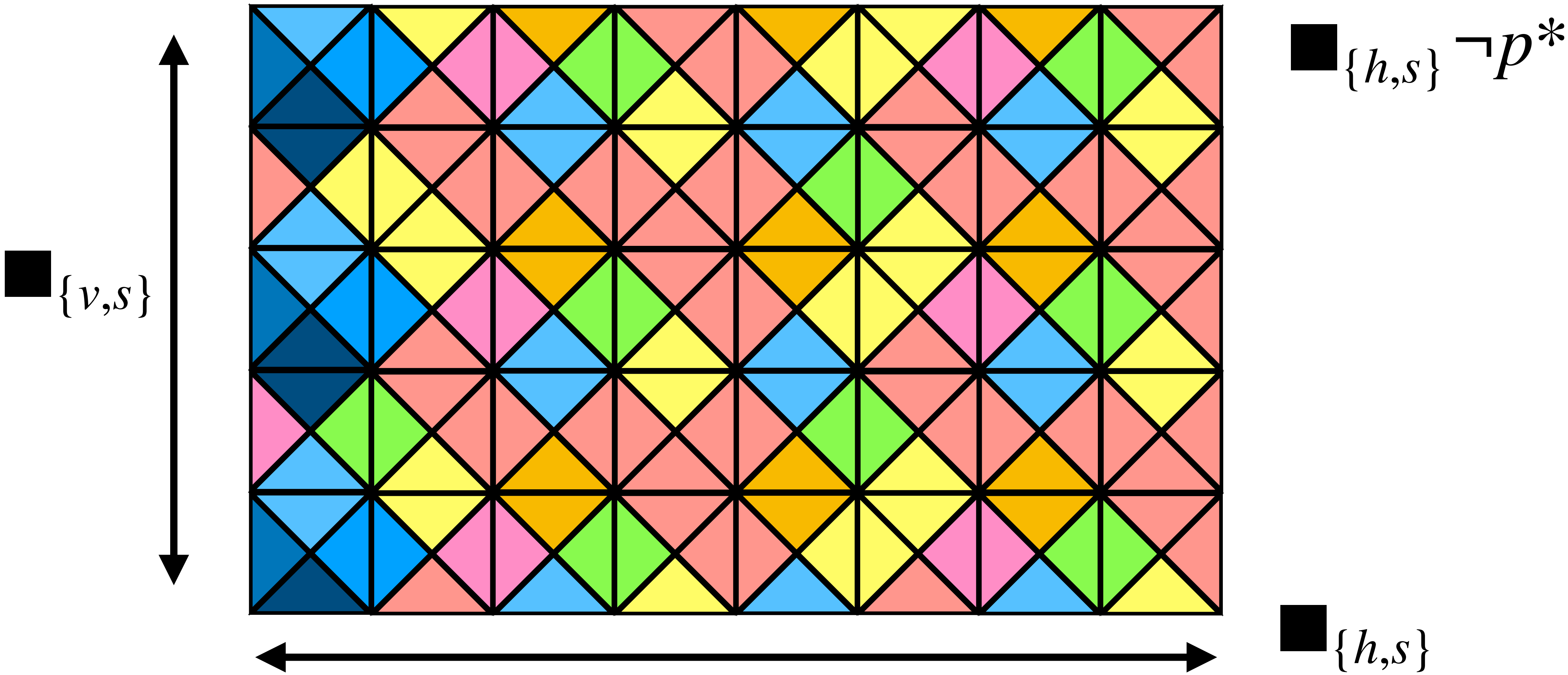
Lemma. If T can tile $\mathbb{N} \times \mathbb{N}$, then Ψ_T is satisfiable

Lemma. If Ψ_T is satisfiable, then T can tile $\mathbb{N} \times \mathbb{N}$

Encoding the Recurring Tile

$$\Psi_T \wedge \blacksquare_{\{v,s\}}[\blacksquare_{\{h,s\}} \neg p^*] \neg \Psi_T$$

T can tile $\mathbb{N} \times \mathbb{N}$ and after removing all rows with the special tile (p^*) we no longer have a tiling



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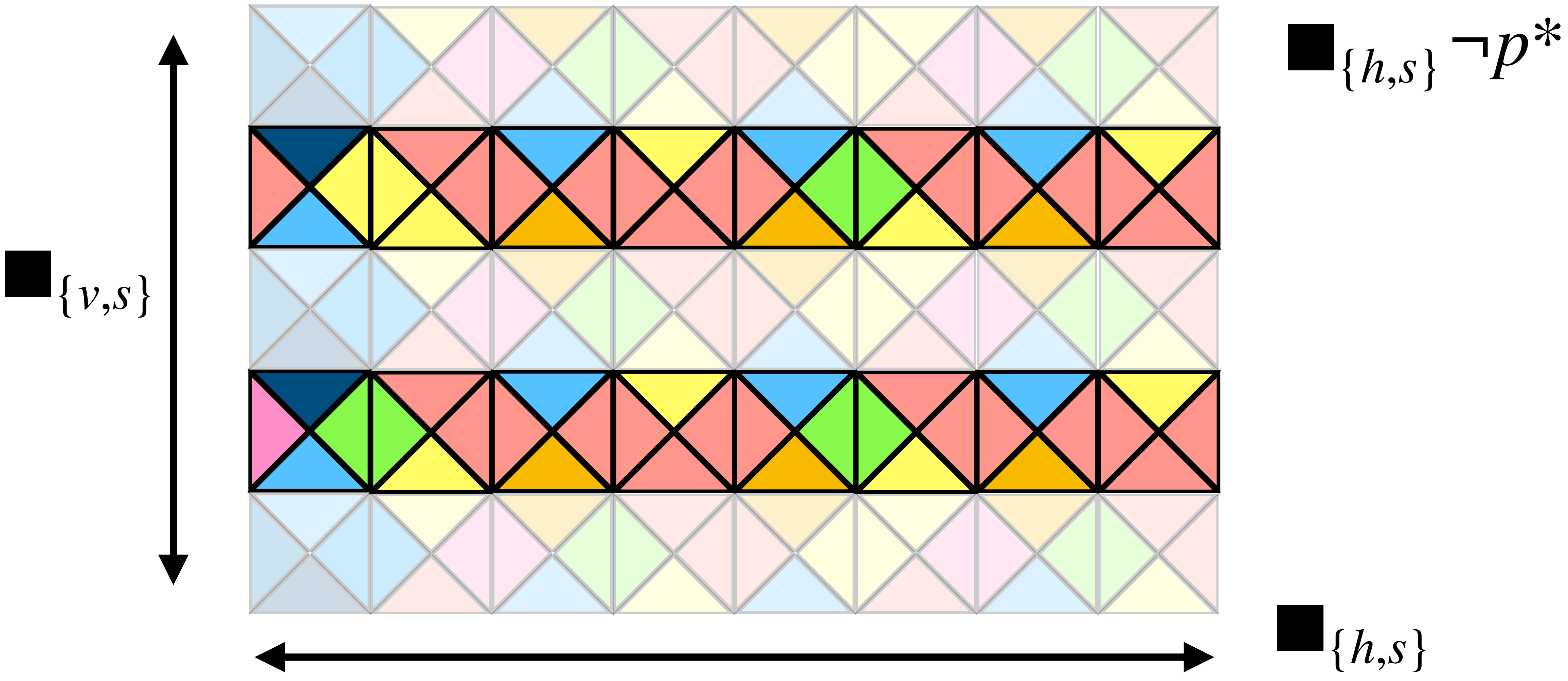

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T can tile $\mathbb{N} \times \mathbb{N}$ and after removing all rows with the special tile (p^*) we no longer have a tiling

Theorem. T can tile $\mathbb{N} \times \mathbb{N}$ with τ^* appearing infinitely often in the first column if and only if $\Psi_T \wedge \blacksquare_{\{v,s\}}[\blacksquare_{\{h,s\}} \neg p^*] \neg \Psi_T$ is satisfiable

Theorem. Satisfiability of APALC is Σ^1_1-hard

Harel. *Effective transformations on infinite trees, with applications to high undecidability, dominoes, and fairness*, 1986.

Part III

Corollaries and Conclusion

Corollaries

Theorem. Satisfiability of APALC is Σ^1_1-hard

Corollary. The set of valid formulas of APALC is neither RE nor co-RE

Open Problem*. Is there a finitary axiomatisation of APAL with common knowledge?

Odifreddi. *Classical recursion theory*, 1989.

Corollaries

Theorem. Satisfiability of APALC is Σ_1^1-hard

Corollary. The set of valid formulas of APALC is neither RE nor co-RE

Open Problem*. Is there a finitary axiomatisation of APAL with common knowledge? **NO!**

Odifreddi. *Classical recursion theory*, 1989.

Letting Agents Do the Work

Group announcement logic (GAL). $\langle G \rangle \varphi$: There is an announcement by agents from group G such that φ is true after the announcement

Coalition announcement logic (CAL). $\langle\!\![G]\!\!\rangle \varphi$: There is an announcement by agents from coalition G such that no matter what agents outside of the coalition announce at the same time, φ is true

Corollary. GALC and CALC do not have finitary axiomatisations

Ågotnes et al. *Group announcement logic*, 2010.

Ågotnes, Van Ditmarsch. *Coalitions and Announcements*, 2008.

Conclusion

	Finitary axiomatisation	Infinitary axiomatisation
APAL	?	✓
GAL	?	✓
CAL	?	?
APALC	✗	✓
GALC	✗	✓
CALC	✗	?

Balbiani et al. *'Knowable' as 'Known After an Announcement'*, 2008.
Ågotnes et al. *Group announcement logic*, 2010.
Ågotnes, Galimullin. *Quantifying over information change with common knowledge*, 2023.