Topic-Based Communication Between Agents*

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Abstract

Communication within groups of agents has been lately the focus of research in dynamic epistemic logic (DEL). This paper studies a recently introduced form of *partial* (more precisely, *topic-based*) *communication*. This type of communication allows for modelling scenarios of multi-agent collaboration and negotiation, and it is particularly well-suited for situations in which sharing all information is not feasible/advisable. The paper can be divided into two parts. In the first part, we present results on invariance and complexity of model checking. Moreover, we compare partial communication with the public announcement and arrow update settings in terms of both language-expressivity and update-expressivity. Regarding the former, the three settings are equivalent, their languages being equally expressive. Regarding the latter, all three modes of communication are incomparable in terms of update-expressivity. In the second part, we shift our attention to *strategic* topic-based communication. We do so by extending the language with a modality that quantifies over the topics the agents can 'talk about', thus allowing a form of *arbitrary partial communication*. For this new framework, we provide a complete axiomatisation, showing also that the new language's model checking problem is *PSPACE*-complete. Finally, we argue that, in terms of expressivity, this new language of arbitrary partial communication is incomparable to that of arbitrary public announcements and also to that of arbitrary arrow updates.

Keywords: Epistemic Logic · Distributed Knowledge · Dynamic Epistemic Logic · Partial Communication · Public Announcements · Arrow Updates · Arbitrary Partial Communication · Arbitrary Public Announcements · Arbitrary Arrow Updates

1 Introduction

Epistemic logic (*EL*; Hintikka 1962) is a powerful framework for representing knowledge/beliefs of both individual agents and groups thereof. When using relational 'Kripke' models, its crucial idea is the use of uncertainty for defining knowledge. Indeed, such structures assign to each agent a binary relation

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indicating *indistinguishability* among epistemic possibilities. Then, it is said that an agent i knows that φ is the case (syntactically: $K_{\mathrm{i}}\,\varphi$) when φ holds in all situations i considers possible. Despite its simplicity, *EL* has become a widespread tool, contributing to the formal study of complex multi-agent epistemic phenomena in philosophy (Hendricks 2006), computer science (Fagin et al. 1995), AI (Meyer and van der Hoek 1995) and economics (de Bruin 2010).

One of the most appealing aspects of *EL* is that it can be used to reason about information change. This has been the main subject of dynamic epistemic logic (*DEL*; van Ditmarsch et al. 2008, van Benthem 2011), a field whose main feature is that actions are semantically represented as operations that transform the underlying semantic model.[1] Within *DEL*, one of the simplest meaningful epistemic actions is that of a *public announcement*: an external source providing the agents with truthful information in a fully public way (Plaza 1989, Gerbrandy and Groeneveld 1997). Yet, the agents do not need to wait for some external entity to feed them with facts: they can also share their individual information with one another. This is arguably a more suitable way of modelling information change in multi-agent (and, in particular, distributed) systems. Agents might occasionally receive information 'from the outside', but the most common form of interaction is the one in which they themselves engage in 'conversations' to share what they have obtained so far. It is this form of information exchange that allows independent entities to engage in collaboration, negotiation, and so on.

Agent communication can take several forms, and some variations have been explored within the *DEL* framework. A single agent might share all her information with everybody, as modelled in Baltag (2010). Alternatively, a group of agents might share all their information only among themselves, as represented by the action of "resolving distributed knowledge" studied in Ågotnes and Wáng (2017). One can even think about this form of communication not as a form of 'sharing', but rather as a form of 'taking' (Baltag and Smets 2020, 2021), which allows the study of public and private forms of reading someone else's information (e.g., hacking).

All these approaches for communication have a common feature: when sharing/taking, the agents share/take *all the available information*. This is of course useful, as then one can reason about the best the agents can do together. But there are also scenarios (arguably more common) in which sharing all her available information might not be feasible or advisable for an agent. For the first, there might be constraints on the communication channels; for the second, agents might not be in a cooperative scenario, but rather in a competitive one. In such cases, one would be rather interested in studying forms of *partial* communication, through which agents share only 'part of what they know'. There might be different ways to make precise what each agent shares, but a natural one is to assume that the 'conversation' is relative to a subject/topic, defined by a given formula χ. Introduced in Velázquez-Quesada (2022), this type of communication allows for more realistic modelling of scenarios of multi-agent collaboration and negotiation.

This paper studies different aspects of this partial communication setting.

[1]This is different, e.g., from *dynamic logic* (Harel et al. 2000), where actions are represented as relations.

It starts (Section 2) by recalling the underlying framework (*EL* with distributed knowledge). Then, it presents the basics of the *partial communication* framework (Section 3), providing definitions (language, semantic interpretation) and results (axiom system, structural equivalence, expressivity and complexity of model checking) as well as comparing it with two well-known *DEL* frameworks, namely public announcements and arrow updates. The comparison shows interesting connections. First, the languages of the three systems are equally expressive.[2] Then, their 'update expressivity' is different. On the one hand, in general, the partial communication and public announcement operations cannot mimic each other: there are scenarios in which, from the language's point of view, the effect of a public announcement cannot be replicated by partial communication, and vice versa. On the other hand, partial communication and arrow updates cannot in general mimic each other either.

Still, in truly competitive scenarios, what matters the most is not the effects of what is being shared, but rather the decision of *what* to share. In other words, what matters is being able to reason about *strategic* topic-based communication. To do so, this paper introduces (Section 4) a logical framework for quantifying over the conversation's topic, thus allowing *arbitrary partial communication*. It presents the basic definitions, providing then results on invariance, axiom system, expressivity and the complexity of its model checking problem. After that, it compares this new setting with that of arbitrary public announcements and that of arbitrary arrow updates. In both cases, it is shown that the languages are, expressivity-wise, incomparable. The paper closes (Section 5) summarising the paper's contents while also discussing further research lines.

2 Background

Models and relative expressivity. Throughout this text, let $\mathtt{A}$ be a finite non-empty group of agents, and let $\mathtt{P}$ be a non-empty enumerable set of atomic propositions.

Definition 2.1 (Model) A *multi-agent relational model* (from now on, a model) is a tuple $M = \langle W, R, V \rangle$ where W (also denoted as $\mathfrak{D}(M)$) is a non-empty set of objects called *possible worlds*, $R = \{R_{\mathtt{i}} \subseteq W \times W \mid \mathtt{i} \in \mathtt{A}\}$ assigns a binary *"indistinguishability"* relation on W to each agent in $\mathtt{A}$ (for $\mathtt{G} \subseteq \mathtt{A}$, define $R_{\mathtt{G}} := \bigcap_{\mathtt{k} \in \mathtt{G}} R_{\mathtt{k}}$), and $V : \mathtt{P} \to \wp(W)$ is an atomic valuation (with $V(p)$ being the set of worlds in M where $p \in \mathtt{P}$ holds). A pair (M, w), where M is a model and $w \in \mathfrak{D}(M)$, is a *pointed model*, with w being the *evaluation point*. A model M is *finite* if and only if both W and $\bigcup_{w \in W} \{p \in \mathtt{P} \mid w \in V(p)\}$ are finite. If $M = \langle W, R, V \rangle$ is finite, its *size* (notation: $|M|$) is given by $|W| + \sum_{\mathtt{i} \in \mathtt{A}} |R_{\mathtt{i}}| + \sum_{w \in W} |\{p \in \mathtt{P} \mid w \in V(p)\}|$.◀

In a model, the agents' indistinguishability relations are arbitrary. In particular, they need to be neither reflexive nor symmetric nor Euclidean nor transitive. There are two reasons for this. On the conceptual side, although equivalence relations are somehow standard for representing the notion of knowledge, several authors have argued against positive and negative introspection, epistemic properties directly connected to the relational properties of

[2] This holds assuming that their epistemic fragment contains the distributed knowledge modality.

transitivity and Euclidicity. Indeed, it has been argued that both forms of introspection are, in many situations, unreachable idealisations that might lead to contradictory situations (see, e.g., Lemmon 1967, Danto 1967, Williamson 2002 for positive introspection, and Hintikka 1962 for negative introspection; see also the discussion in the introduction of Fervari and Velázquez-Quesada 2019). On the technical side, the partial communication operation (Definition 3.1 below) preserves reflexivity but neither transitivity nor Euclidicity. Thus, requiring the two latter properties would have made the operation 'non-suitable', as it would change the class of models.[3] If needed, asking for the relations to be reflexive is (both conceptually and technically) a safe choice. This paper takes rather a more general perspective, working with arbitrary relations. Because of this, *"knowledge"* here is neither truthful nor positively/negatively introspective. It rather corresponds, simply, to "what is true in all the agent's (agents') epistemic alternatives".

Definition 2.2 (Relative expressivity) Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two languages interpreted over pointed models. It is said that $\mathcal{L}_2$ *is at least as expressive as* $\mathcal{L}_1$ (notation: $\mathcal{L}_1 \preccurlyeq \mathcal{L}_2$) if and only if for every $\phi_1 \in \mathcal{L}_1$ there is $\phi_2 \in \mathcal{L}_2$ such that ϕ_1 and ϕ_2 have the same truth-value in every pointed model (i.e., $(M, w) \Vdash \phi_1$ if and only if $(M, w) \Vdash \phi_2$ for every M and every $w \in \mathfrak{D}(M)$). Write $\mathcal{L}_1 \approx \mathcal{L}_2$ when $\mathcal{L}_1 \preccurlyeq \mathcal{L}_2$ and $\mathcal{L}_2 \preccurlyeq \mathcal{L}_1$; write $\mathcal{L}_1 \prec \mathcal{L}_2$ when $\mathcal{L}_1 \preccurlyeq \mathcal{L}_2$ and $\mathcal{L}_2 \not\preccurlyeq \mathcal{L}_1$; write $\mathcal{L}_1 \asymp \mathcal{L}_2$ when $\mathcal{L}_1 \not\preccurlyeq \mathcal{L}_2$ and $\mathcal{L}_2 \not\preccurlyeq \mathcal{L}_1$. ◀

Note: for proving $\mathcal{L}_1 \not\preccurlyeq \mathcal{L}_2$, it is enough to find two (classes of) pointed models that agree on all formulas in $\mathcal{L}_2$ but can be distinguished by a formula in $\mathcal{L}_1$. Indeed, let (N, u) and (N', u') be the pointed models indistinguishable by $\mathcal{L}_2$ and let $\phi_1 \in \mathcal{L}_1$ be a formula that distinguishes them. For a contradiction, suppose $\mathcal{L}_1 \preccurlyeq \mathcal{L}_2$. Then, there would be a formula $\phi_2 \in \mathcal{L}_2$ agreeing with ϕ_1 in every pointed model; in particular, they would agree in both (N, u) and (N', u'). But (N, u) and (N', u') cannot be distinguished by $\mathcal{L}_2$, so ϕ_2 has the same truth value in both pointed models. Then, so does ϕ_1, contradicting the fact that it can distinguish them.

Syntax and semantics. Here is this paper's basic language for describing pointed models.

Definition 2.3 (Language $\mathcal{L}$) Formulas φ, ψ in $\mathcal{L}$ are given by

$$\varphi, \psi ::= p \mid \neg\varphi \mid (\varphi \land \psi) \mid \mathrm{D}_{\mathtt{G}}\, \varphi$$

for $p \in \mathtt{P}$ and $\varnothing \subset \mathtt{G} \subseteq \mathtt{A}$. Boolean constants and other Boolean operators are defined as usual. We will also omit parentheses whenever it does not impede clarity. Define also $\mathrm{K}_{\mathtt{i}}\, \varphi := \mathrm{D}_{\{\mathtt{i}\}}\, \varphi$ and $\widehat{\mathrm{K}}_{\mathtt{i}}\, \varphi := \neg \mathrm{K}_{\mathtt{i}} \neg\varphi$. The *set of atoms* in a formula is defined recursively as usual:

$$\mathrm{at}(p) := \{p\}, \quad \mathrm{at}(\neg\varphi) := \mathrm{at}(\varphi), \quad \mathrm{at}(\varphi \land \psi) := \mathrm{at}(\varphi) \cup \mathrm{at}(\psi), \quad \mathrm{at}(\mathrm{D}_{\mathtt{G}}\, \varphi) := \mathrm{at}(\varphi).$$

Finally, the *size* of φ, denoted as $|\varphi|$, is defined recursively in the standard way:

$$|p| := 1, \quad |\neg\varphi| := |\varphi| + 1, \quad |\varphi \land \psi| := |\varphi| + |\psi| + 1, \quad |\mathrm{D}_{\mathtt{G}}\, \varphi| := |\varphi| + 1. \quad ◀$$

[3] Moreover, rule $\mathsf{RE}_{\mathtt{S}:\chi}$ in Table 2 would not preserve validity.

The language $\mathcal{L}$ contains a modality $\mathrm{D}_{\mathtt{G}}$ for each non-empty group of agents $\mathtt{G} \subseteq \mathtt{A}$. Formulas of the form $\mathrm{D}_{\mathtt{G}}\,\varphi$ are read as "the agents in $\mathtt{G}$ know φ distributively"; thus, $\mathrm{K}_{\mathtt{i}}\,\varphi$ is read as "$\mathtt{i}$ knows φ distributively", i.e., "agent $\mathtt{i}$ knows φ".

The use of the modality for distributed knowledge (Hilpinen 1977, Halpern and Moses 1984, 1985, 1990) might require further justification. Intuitively, φ is distributed knowledge among a group of agents if and only if it follows from the combination of the individual knowledge of the group's members (or, in other words, if the agents *would* know φ by putting all their information together). Distributed knowledge thus 'pre-encodes' what a group of agents would know if they were to share their individual information among themselves. Because of this, it will be a very useful tool in this text.

Now, in models that represent directly the individual knowledge of the agents, distributed knowledge has a straightforward definition: put the knowledge of the members of the group together (using the union operation), and then get the closure under logical consequence. In relational models, which represent rather the agent's uncertainty, there is also a natural way of defining a relation for the agents' distributed knowledge: given a world w, the group $\mathtt{G}$ will consider u as possible if and only if, given w, everybody in $\mathtt{G}$ considers u possible (or, equivalently, *no one* in $\mathtt{G}$ can rule u out). In other words, the indistinguishability relation for the distributed knowledge of a group is the intersection of the indistinguishability relations of the group's members. With this, the language's semantic interpretation is as follows.

Definition 2.4 (Semantic interpretation for $\mathcal{L}$) Let (M, w) be a pointed model with $M = \langle W, R, V\rangle$. The satisfiability relation $\Vdash$ between (M, w) and formulas in $\mathcal{L}$ is defined inductively.

$$
\begin{array}{lll}
(M,w) \Vdash p & \text{iff}_{def} & w \in V(p), \\
(M,w) \Vdash \neg\varphi & \text{iff}_{def} & (M,w) \nVdash \varphi, \\
(M,w) \Vdash \varphi \wedge \psi & \text{iff}_{def} & (M,w) \Vdash \varphi \text{ and } (M,w) \Vdash \psi, \\
(M,w) \Vdash \mathrm{D}_{\mathtt{G}}\,\varphi & \text{iff}_{def} & \text{for all } u \in W, \text{ if } R_{\mathtt{G}}wu \text{ then } (M,u) \Vdash \varphi.
\end{array}
$$

Given a model M and a formula φ,

- the set $[\![\varphi]\!]^M := \{w \in \mathfrak{D}(M) \mid (M,w) \Vdash \varphi\}$ contains the worlds in $\mathfrak{D}(M)$ in which φ holds (also called φ-worlds);
- the (note: equivalence) relation

$$\sim_{\varphi}^{M} := ([\![\varphi]\!]^M \times [\![\varphi]\!]^M) \cup ([\![\neg\varphi]\!]^M \times [\![\neg\varphi]\!]^M)$$

  splits $\mathfrak{D}(M)$ into (up to) two equivalence classes: one containing all φ-worlds, and the other containing all $\neg\varphi$-worlds.

A formula φ is valid (notation: $\Vdash \varphi$) if and only if $(M, w) \Vdash \varphi$ for every $w \in \mathfrak{D}(M)$ of every model M. ◀

Axiom system. The axiom system L (Table 1; Halpern and Moses 1990, Fagin et al. 1992) characterises the valid formulas in $\mathcal{L}$. The behaviour of Boolean operators is taken care of by PR and MP. For the modality $\mathrm{D}_{\mathtt{G}}$, while rule G_{D} indicates that it 'contains' all validities, axiom K_{D} indicates that it is closed

PR: $\vdash \varphi$ for φ a propositionally valid scheme	
MP: If $\vdash \varphi$ and $\vdash \varphi \to \psi$ then $\vdash \psi$	
$\mathsf{K_D}$: $\vdash \mathrm{D}_{\mathtt{G}}(\varphi \to \psi) \to (\mathrm{D}_{\mathtt{G}}\,\varphi \to \mathrm{D}_{\mathtt{G}}\,\psi)$	$\mathsf{G_D}$: If $\vdash \varphi$ then $\vdash \mathrm{D}_{\mathtt{G}}\,\varphi$
$\mathsf{M_D}$: $\vdash \mathrm{D}_{\mathtt{G}}\,\varphi \to \mathrm{D}_{\mathtt{G}'}\,\varphi$ for $\mathtt{G} \subseteq \mathtt{G}'$	

Table 1: Axiom system L.

under modus ponens, and axiom $\mathsf{M_D}$ states that it is monotone on the group of agents (if φ is distributively known by $\mathtt{G}$, then it is also distributively known by any larger group $\mathtt{G}'$).

Theorem 1 *The axiom system* L *(Table 1) is sound and strongly complete for $\mathcal{L}$ w.r.t. the given class of models.* ■

Structural equivalence. When discussing the expressivity of a language, it is useful to have a semantic notion that connects two pointed models when they cannot be distinguished by the language's formulas. For the basic modal language (Boolean operators plus modalities for the individual relations), the notion of bisimulation plays this role (see, e.g., Blackburn et al. 2001, Definition 2.18 and Theorem 2.20). When the modality for distributed knowledge is included, one needs rather the notion of *collective* bisimulation (Roelofsen 2007), which expands on a standard bisimulation by asking for the **forth** and **back** clauses to be satisfied not only by all singletons $\{\mathtt{i}\} \subseteq \mathtt{A}$ but also by all groups of agents $\mathtt{G} \subseteq \mathtt{A}$. The definition provided below makes a further generalisation, making the relevant set of atoms a parameter. This will be useful for discussing the expressivity of languages that quantify over information change (Section 4).

Definition 2.5 (Collective Q-bisimulation) Let $\mathtt{Q} \subseteq \mathtt{P}$ be a set of atoms; let $M = \langle W, R, V\rangle$ and $M' = \langle W', R', V'\rangle$ be two models. A non-empty relation $Z \subseteq W \times W'$ is a *collective* $\mathtt{Q}$*-bisimulation between M and M'* if and only if every $(u, u') \in Z$ satisfies the following.

- **Atoms**. For every $p \in \mathtt{Q}$: $u \in V(p)$ if and only if $u' \in V'(p)$.
- **Forth**. For every $\mathtt{G} \subseteq \mathtt{A}$ and every $v \in W$: if $R_{\mathtt{G}}uv$ then there is $v' \in W'$ such that both $R'_{\mathtt{G}}u'v'$ and $(v, v') \in Z$.
- **Back**. For every $\mathtt{G} \subseteq \mathtt{A}$ and every $v' \in W'$: if $R'_{\mathtt{G}}u'v'$ then there is $v \in W$ such that both $R_{\mathtt{G}}uv$ and $(v, v') \in Z$.

Write $M \rightleftarrows^{\mathtt{Q}}_{C} M'$ if and only if there is a collective $\mathtt{Q}$-bisimulation between M and M'. Write $(M, w) \rightleftarrows^{\mathtt{Q}}_{C} (M', w')$ if and only if a witness for $M \rightleftarrows^{\mathtt{Q}}_{C} M'$ contains the pair (w, w'). If $\mathtt{Q}$ is the full set of atoms $\mathtt{P}$, it will be omitted from the notation.◀

Note that the relation of collective $\mathtt{Q}$-bisimilarity is an equivalence relation, both on models and pointed models.[4]

The following proposition shows that a collective bisimulation is useful for our purposes: the language $\mathcal{L}$ is invariant under collective bisimilarity.

[4]Indeed, take arbitrary pointed models (M, w), (M', w') and (M'', w''). Then, ***(i)*** the identity relation on W is a witness for both $M \rightleftarrows^{\mathtt{Q}}_{C} M$ and $(M, w) \rightleftarrows^{\mathtt{Q}}_{C} (M, w)$; ***(ii)*** if $Z \subseteq W \times W'$ is a witness for $M \rightleftarrows^{\mathtt{Q}}_{C} M'$ (resp., $(M, w) \rightleftarrows^{\mathtt{Q}}_{C} (M', w')$), then $Z^{-1} \subseteq W' \times W$ is a witness for $M' \rightleftarrows^{\mathtt{Q}}_{C} M$ (resp.,

Theorem 2 ($\rightleftarrows_C$ implies $\mathcal{L}$-equivalence) *Let (M, w) and (M', w') be two pointed models; take $\mathtt{Q} \subseteq \mathtt{P}$. If $(M, w) \rightleftarrows_C^{\mathtt{Q}} (M', w')$ then, for every $\psi \in \mathcal{L}$ with $\mathsf{at}(\psi) \subseteq \mathtt{Q}$,*

$$(M, w) \Vdash \psi \quad \textit{if and only if} \quad (M', w') \Vdash \psi.$$

Proof. Proofs showing that a form of structural equivalence implies invariance for a language usually proceed by structural induction on the language's formulas.[5] For the case of collective bisimilarity and $\mathcal{L}$, see Roelofsen (2007). ■

Model checking The complexity of the model checking problem for $\mathcal{L}$ (given a pointed model and a formula in $\mathcal{L}$, decide whether the formula is true at the pointed model) is in *P* Fagin et al. (1995, Page 67).

3 Partial communication

The intuition behind the action of partial communication is that, through it, a group of agents $\mathtt{S} \subseteq \mathtt{A}$ share, with everybody, all their information about a given topic χ. Before looking at its formal definition, it is useful to consider the definition of a simpler action: one through which the agents in $\mathtt{S}$ share *all their information* with everybody.[6]

After agents in $\mathtt{S}$ share all their information with everybody, a given agent $\mathtt{i}$ at a world w will consider a world u possible if and only if neither her nor any agent in $\mathtt{S}$ could rule out u from w before the action. In other words, after this full communication action, an agent $\mathtt{i}$ will consider a world u possible from a world w if and only if she and every agent in $\mathtt{S}$ already considered u possible from w. This means that, after the action, $\mathtt{i}$'s indistinguishability relation is the *intersection* of the relations $R_{\mathtt{i}}$ and $R_{\mathtt{S}}$: edges that are not labelled by all communicating agents will be removed.

Now, suppose the agents in $\mathtt{S}$ share only 'their information about χ' (intuitively, only what has allowed them to distinguish between χ- and $\neg\chi$-worlds). In such case, as argued in Velázquez-Quesada (2022), edges between worlds agreeing in χ's truth-value are not 'part of the discussion', and thus they should not be eliminated. In other words, only edges connecting worlds disagreeing in χ's truth-value can be eliminated, and they will be eliminated if and only if they are not labelled by all communicating agents.

3.1 Syntax, semantics, and model checking

Definition 3.1 (Partial communication) Let $M = \langle W, R, V \rangle$ be a model; take a group of agents $\mathtt{S} \subseteq \mathtt{A}$ and a formula χ. The model $M_{\mathtt{S}:\chi} = \langle W, R^{\mathtt{S}:\chi}, V \rangle$, the result of agents in $\mathtt{S}$ sharing all they know about χ with everybody, is such that

$(M', w') \rightleftarrows_C^{\mathtt{Q}} (M, w)$); ***(iii)*** if $Z_1 \subseteq W \times W'$ and $Z_2 \subseteq W' \times W''$ are witnesses for $M \rightleftarrows_C^{\mathtt{Q}} M'$ and $M' \rightleftarrows_C^{\mathtt{Q}} M''$ (resp., $(M, w) \rightleftarrows_C^{\mathtt{Q}} (M', w')$ and $(M', w') \rightleftarrows_C^{\mathtt{Q}} (M'', w'')$), then $Z_2 \circ Z_1$ is a witness for $M \rightleftarrows_C^{\mathtt{Q}} M''$ (resp., $(M, w) \rightleftarrows_C^{\mathtt{Q}} (M'', w'')$).

[5] The proofs typically start by pulling out the universal quantifier over formulas. This way, the statement becomes *"for every ψ (containing only atoms from $\mathtt{Q}$), any structurally equivalent pointed models agree on ψ's truth-value"*. This yields a stronger inductive hypothesis (IH) thanks to which the proof can go through. This will be done throughout the rest of the text.

[6] Cf. the *resolving* action of Ågotnes and Wáng (2017), through which a group of agents share all their information *within themselves*.

$$R^{\mathsf{S}:\chi}{}_{\mathtt{i}} := R_{\mathtt{i}} \cap (R_{\mathsf{S}} \cup \sim^{M}_{\chi}).$$ ◀

A couple of observations are useful.

- First, the indistinguishability relation for the *distributed knowledge* of a group of agents $\mathtt{G}$ in the new model (i.e., the intersection of the new indistinguishability relations), denoted as $R^{\mathsf{S}:\chi}{}_{\mathtt{G}}$, can be written in a slightly simplified way:

$$R^{\mathsf{S}:\chi}{}_{\mathtt{G}} = \bigcap_{\mathtt{i}\in\mathtt{G}} R^{\mathsf{S}:\chi}{}_{\mathtt{i}} = \bigcap_{\mathtt{i}\in\mathtt{G}} (R_{\mathtt{i}} \cap (R_{\mathsf{S}} \cup \sim^{M}_{\chi})) = R_{\mathtt{G}} \cap (R_{\mathsf{S}} \cup \sim^{M}_{\chi}) = R_{\mathtt{G}\cup\mathsf{S}} \cup (R_{\mathtt{G}} \cap \sim^{M}_{\chi}).$$

- Second: $R^{\varnothing:\chi}{}_{\mathtt{i}} = R_{\mathtt{i}}$. Thus, if the set of communicating agents S is empty, indistinguishability (and hence knowledge) remains the same.

On the syntactic side, the following modality is useful for describing the effects of the action of partial communication.

Definition 3.2 (Modality $[\mathsf{S}:\chi]$ and language $\mathcal{L}_{\mathsf{PC}}$) Define $\mathcal{L}_{\mathsf{PC}}[0] := \mathcal{L}$, where PC stands for *partial communication*. Then, define $\mathcal{L}_{\mathsf{PC}}[i{+}1]$ as the result of extending $\mathcal{L}_{\mathsf{PC}}[i]$ with an additional modality $[\mathsf{S}:\chi]$ for $\mathsf{S} \subseteq \mathtt{A}$ and $\chi \in \mathcal{L}_{\mathsf{PC}}[i]$. The language $\mathcal{L}_{\mathsf{PC}}$ is the union of all $\mathcal{L}_{\mathsf{PC}}[n]$ with $n \in \mathbb{N}$, thus essentially extending $\mathcal{L}$ with a modality $[\mathsf{S}:\chi]$ for each $\mathsf{S} \subseteq \mathtt{A}$ and each formula χ. The set of atoms and size for formulas in $\mathcal{L}_{\mathsf{PC}}$ is as in Definition 2.3 with the additional clauses $\mathrm{at}([\mathsf{S}:\chi]\,\varphi) := \mathrm{at}(\chi) \cup \mathrm{at}(\varphi)$ and $|[\mathsf{S}:\chi]\,\varphi| := |\chi| + |\varphi| + 1$, respectively. For the semantic interpretation,

$$(M, w) \Vdash [\mathsf{S}:\chi]\,\varphi \quad \text{iff}_{def} \quad (M_{\mathsf{S}:\chi}, w) \Vdash \varphi.$$

Define $\langle \mathsf{S}:\chi \rangle\,\varphi := \neg\,[\mathsf{S}:\chi]\,\neg\varphi$. Note how this implies $\Vdash \langle \mathsf{S}:\chi \rangle\,\varphi \leftrightarrow [\mathsf{S}:\chi]\,\varphi$. ◀

Further motivation and details on the partial communication setting can be found in Velázquez-Quesada (2022). Still, here are two properties that help to understand what the action does.

- If $\Vdash \chi_1 \leftrightarrow \chi_2$ then $\Vdash [\mathsf{S}:\chi_1]\,\varphi \leftrightarrow [\mathsf{S}:\chi_2]\,\varphi$: logically equivalent topics have the same communication effect.

- $\Vdash [\mathsf{S}:\chi]\,\varphi \leftrightarrow [\mathsf{S}:\neg\chi]\,\varphi$: communication on a topic is just as communication on its negation.

Finally, note that partial communication is *not* a generalisation of an action through which some agents share *all* their information. For this to be the case, the "some agents share all" action should be a particular instance of the partial communication setting, and this is not the case: there is no formula χ such that, in every possible situation, communication about χ is equivalent to communication about all topics.

Axiom system. The axioms and rule of Table 2 form, together with those in Table 1, a sound and strongly complete axiom system for $\mathcal{L}_{\mathsf{PC}}$. They rely on the *DEL* reduction axioms technique (for an explanation, see Wang and Cao 2013 or van Ditmarsch et al. 2008, Section 7.4), which, in turn, crucially relies on the existence of a (recursively defined) truth-preserving translation from $\mathcal{L}_{\mathsf{PC}}$ to $\mathcal{L}$. In the translation, axiom $\mathsf{A}^{\mathsf{D}}_{\mathsf{S}:\chi}$ is the central one, as it characterises

$\mathsf{A}^{p}_{\mathsf{S}:\chi}$:	$\vdash [\mathsf{S}\colon\chi]\, p \leftrightarrow p$
$\mathsf{A}^{\neg}_{\mathsf{S}:\chi}$:	$\vdash [\mathsf{S}\colon\chi]\, \neg\varphi \leftrightarrow \neg\, [\mathsf{S}\colon\chi]\, \varphi$
$\mathsf{A}^{\wedge}_{\mathsf{S}:\chi}$:	$\vdash [\mathsf{S}\colon\chi](\varphi \wedge \psi) \leftrightarrow ([\mathsf{S}\colon\chi]\, \varphi \wedge [\mathsf{S}\colon\chi]\, \psi)$
$\mathsf{A}^{\mathrm{D}}_{\mathsf{S}:\chi}$:	$\vdash [\mathsf{S}\colon\chi]\, \mathrm{D}_{\mathsf{G}}\, \varphi \leftrightarrow (\mathrm{D}_{\mathsf{S}\cup\mathsf{G}}\, [\mathsf{S}\colon\chi]\, \varphi \wedge \mathrm{D}^{\chi}_{\mathsf{G}}\, [\mathsf{S}\colon\chi]\, \varphi)$
$\mathsf{RE}_{\mathsf{S}:\chi}$:	If $\vdash \varphi_1 \leftrightarrow \varphi_2$ then $\vdash [\mathsf{S}\colon\chi]\, \varphi_1 \leftrightarrow [\mathsf{S}\colon\chi]\, \varphi_2$

Table 2: Additional axioms and rules for $\mathsf{L}_{\mathsf{S}:\chi}$.

distributed knowledge after the operation in terms of distributed knowledge about the effects of the operation. Using the abbreviation

$$\mathrm{D}^{\chi}_{\mathsf{G}}\, \varphi := (\chi \rightarrow \mathrm{D}_{\mathsf{G}}(\chi \rightarrow \varphi)) \wedge (\neg\chi \rightarrow \mathrm{D}_{\mathsf{G}}(\neg\chi \rightarrow \varphi))$$

(*"agents in* G *know distributively that* χ*'s truth value, regardless of what it is, implies* φ*"*),

the axiom indicates that a group G knows φ distributively after the action ($[\mathsf{S}\colon\chi]\, \mathrm{D}_{\mathsf{G}}\, \varphi$) if and only if the group $\mathsf{S} \cup \mathsf{G}$ knew, distributively, that φ would hold after the action ($\mathrm{D}_{\mathsf{S}\cup\mathsf{G}}\, [\mathsf{S}\colon\chi]\, \varphi$) and the agents in G know distributively that χ's truth-value, regardless of what it is, implies that the action will make φ true ($\mathrm{D}^{\chi}_{\mathsf{G}}\, [\mathsf{S}\colon\chi]\, \varphi$).

From these axioms and rule (Table 2) together with their induced translation (see Velázquez-Quesada 2022 for details), the following theorem follows.

Theorem 3 *The axiom system* $\mathsf{L}_{\mathsf{S}:\chi}$ *(Table 1+Table 2) is sound and strongly complete for* $\mathcal{L}_{\mathsf{PC}}$ *.*

So far this section has recalled basic definitions and results from the partial communication setting. The following results on structural equivalence, expressivity and complexity, are new.

Structural equivalence. As it turns out, the partial communication modality $[\mathsf{S}\colon\chi]$ (and thus, from Theorem 2, the full language $\mathcal{L}_{\mathsf{PC}}$) is invariant under collective bisimilarity.

Theorem 4 ($\rightleftarrows_C$ implies $\mathcal{L}_{\mathsf{PC}}$ -equivalence) *Let* (M, w) *and* (M', w') *be two pointed models; take* $\mathsf{Q} \subseteq \mathsf{P}$*. If* $(M, w) \rightleftarrows^{\mathsf{Q}}_{C} (M', w')$ *then, for every* $\psi \in \mathcal{L}_{\mathsf{PC}}$ *with* $\mathsf{at}(\psi) \subseteq \mathsf{Q}$*,*

$$(M, w) \Vdash \psi \quad \textit{if and only if} \quad (M', w') \Vdash \psi.$$

Proof. The language $\mathcal{L}_{\mathsf{PC}}$ is the union of $\mathcal{L}_{\mathsf{PC}}[n]$ for all $n \in \mathbb{N}$, so the proof proceeds by induction on n. In fact, the text proves a stronger statement: for every $\psi \in \mathcal{L}_{\mathsf{PC}}$ with $\mathsf{at}(\psi) \subseteq \mathsf{Q}$ and every (M, w) and (M', w'), if $(M, w) \rightleftarrows^{\mathsf{Q}}_{C} (M', w')$ then ***(1)*** $(M, w) \Vdash \psi$ if and only if $(M', w') \Vdash \psi$, and **(2)** $(M_{\mathsf{S}:\psi}, w) \rightleftarrows^{\mathsf{Q}}_{C} (M'_{\mathsf{S}:\psi}, w')$. Details can be found in the appendix. ■

Expressivity. It is clear that $\mathcal{L} \preccurlyeq \mathcal{L}_{\mathsf{PC}}$, as every formula in the former is also in the latter. Moreover: the reduction axioms in Table 2 define a (recursive) translation $tr : \mathcal{L}_{\mathsf{PC}} \rightarrow \mathcal{L}$ such that $\varphi \in \mathcal{L}_{\mathsf{PC}}$ implies $\Vdash \varphi \leftrightarrow tr(\varphi)$ (for details, see Velázquez-Quesada 2022).[7] This implies $\mathcal{L}_{\mathsf{PC}} \preccurlyeq \mathcal{L}$ and thus $\mathcal{L} \approx \mathcal{L}_{\mathsf{PC}}$: the languages $\mathcal{L}$ and $\mathcal{L}_{\mathsf{PC}}$ are equally expressive.

Model checking Now we address the complexity of the model checking problem for $\mathcal{L}_{\mathsf{PC}}$ by providing an algorithm that works in polynomial time (in the sizes of an input model and formula). In particular, we interested in the *global* model checking problem.

Definition 3.3 Given a finite model $M = \langle W, R, V \rangle$ and a formula $\varphi \in \mathcal{L}_{\mathsf{PC}}$, the *global model checking problem* for $\mathcal{L}_{\mathsf{PC}}$ consists in finding all $w \in W$ such that $(M, w) \Vdash \varphi$. ◀

Given a finite pointed model (M, w) and a formula $\varphi \in \mathcal{L}_{\mathsf{PC}}$, the model checking strategy uses an ordered list containing the subformulas that need to be evaluated for deciding whether φ holds at (M, w). Intuitively, the ordering allows the algorithm to deal with formulas *inside* communication modalities (i.e., the χ's in $[\mathsf{S}\colon\chi]\,\psi$) before dealing with formulas *within the scope* of such modalities (i.e., the ψ's in $[\mathsf{S}\colon\chi]\,\psi$). In this way, when $[\mathsf{S}\colon\chi]\,\psi$ needs to be evaluated, the effects of $[\mathsf{S}\colon\chi]$ on the model are already known.

To obtain such a list, use a strategy similar to that in Kuijer (2015). Start by creating the set $\mathrm{subm}(\varphi, \epsilon)$, which contains all subformulas and partial communication modalities in φ, taking additional care of labelling all these expressions with the sequence α of partial communication modalities inside the scope of which they appear (here, ϵ is the empty string). Using "$\cdot$" for concatenation, the function subm is recursively defined as

$$\begin{aligned}
\mathrm{subm}(p, \alpha) &:= \{p^{\alpha}\} \\
\mathrm{subm}(\neg\varphi, \alpha) &:= \{(\neg\varphi)^{\alpha}\} \cup \mathrm{subm}(\varphi, \alpha) \\
\mathrm{subm}(\varphi \land \psi, \alpha) &:= \{(\varphi \land \psi)^{\alpha}\} \cup \mathrm{subm}(\varphi, \alpha) \cup \mathrm{subm}(\psi, \alpha) \\
\mathrm{subm}(\mathrm{D}_{\mathsf{G}}\,\varphi, \alpha) &:= \{(\mathrm{D}_{\mathsf{G}}\,\varphi)^{\alpha}\} \cup \mathrm{subm}(\varphi, \alpha) \\
\mathrm{subm}([\mathsf{S}\colon\chi]\,\varphi, \alpha) &:= \{([\mathsf{S}\colon\chi]\,\varphi)^{\alpha}, [\mathsf{S}\colon\chi]^{\alpha}\} \cup \mathrm{subm}(\chi, \alpha) \cup \mathrm{subm}(\varphi, \alpha\cdot[\mathsf{S}\colon\chi])
\end{aligned}$$

As an example, consider the formula $[\mathsf{S}_1\colon p \land q]\,[\mathsf{S}_2\colon q]\,\mathrm{D}_{\mathsf{G}}\,p$. According to the definition above, the set $\mathrm{subm}([\mathsf{S}_1\colon p \land q]\,[\mathsf{S}_2\colon q]\,\mathrm{D}_{\mathsf{G}}\,p, \epsilon)$ is given by

$$\left\{ \begin{array}{c} [\mathsf{S}_1\colon p \land q]\,[\mathsf{S}_2\colon q]\,\mathrm{D}_{\mathsf{G}}\,p,\ [\mathsf{S}_1\colon p \land q],\ p \land q,\ p,\ q,\ ([\mathsf{S}_2\colon q]\,\mathrm{D}_{\mathsf{G}}\,p)^{[\mathsf{S}_1:p\land q]}, \\ [\mathsf{S}_2\colon q]^{[\mathsf{S}_1:p\land q]},\ q^{[\mathsf{S}_1:p\land q]},\ (\mathrm{D}_{\mathsf{G}}\,p)^{[\mathsf{S}_1:p\land q]\,[\mathsf{S}_2:q]},\ p^{[\mathsf{S}_1:p\land q]\,[\mathsf{S}_2:q]} \end{array} \right\}$$

Then, obtain the required ordered list by ordering the elements of $\mathrm{subm}(\varphi, \epsilon)$ in the following way: for $\psi_1^{\sigma}, \psi_2^{\tau}$ (with σ and τ the labellings)[8] we have that ψ_1^{σ} precedes ψ_2^{τ} if and only if

- ψ_1^{σ} and ψ_2^{τ} appear within some modalities $[\mathsf{S}\colon\chi]$, and $\sigma < \tau$,[9] or else
- ψ_1^{σ} appears within some $[\mathsf{S}\colon\chi]$, and ψ_2^{τ} does not, or else

[7] Note: the translation's complexity might be exponential, as it is for similar *DEL*s (e.g., public announcement: Lutz 2006).

[8] We would like to reiterate that since we include in $\mathrm{subm}(\varphi, \epsilon)$ not only subformulas of φ but modalities $[\mathsf{S}\colon\chi]$ appearing in φ as well, elements ψ_1^{σ} and ψ_2^{τ} are not necessarily formulas.

[9] That is, σ is a proper prefix of τ.

- ψ_1^σ is some modality $[\mathsf{S}\colon\chi]$, ψ_2^τ is not, and $\sigma < \tau$, or else
- neither ψ_1^σ nor ψ_2^τ appear in some modalities $[\mathsf{S}\colon\chi]$, and $\tau < \sigma$, or else
- both ψ_1^σ are ψ_2^τ are some modalities $[\mathsf{S}\colon\chi]$, and $\sigma < \tau$, or else
- $\sigma = \tau$, and ψ_1^σ is either a subformula of ψ_2^τ or is a modality appearing in ψ_2^τ, or else
- ψ_1 appears to the left of χ in φ.

As an example, ordering the elements of $\text{subm}([\mathsf{S}_1\colon p \wedge q]\,[\mathsf{S}_2\colon q]\,\mathrm{D}_{\mathsf{G}}\,p, \epsilon)$ yields

$$p\,,\; q\,,\; p \wedge q\,,\; [\mathsf{S}_1\colon p \wedge q]\,,\; q^{[\mathsf{S}_1\colon p\wedge q]}\,,\; [\mathsf{S}_2\colon q]^{[\mathsf{S}_1\colon p\wedge q]}\,,\; p^{[\mathsf{S}_1\colon p\wedge q]\,[\mathsf{S}_2\colon q]}\,,\; (\mathrm{D}_{\mathsf{G}}\,p)^{[\mathsf{S}_1\colon p\wedge q]\,[\mathsf{S}_2\colon q]}\,,\; ([\mathsf{S}_2\colon q]\,\mathrm{D}_{\mathsf{G}}\,p)^{[\mathsf{S}_1\colon p\wedge q]}\,,\; [\mathsf{S}_1\colon p \wedge q]\,[\mathsf{S}_2\colon q]\,\mathrm{D}_{\mathsf{G}}\,p$$

Note how, for a given formula φ, the number of elements in $\text{subm}(\varphi)$ (the subformulas and partial communication modalities in φ) is bounded by $O(|\varphi|)$.

Once the list $\text{subm}(\varphi)$ is ready, run the labelling Algorithm 1, which is inspired by the model checking procedure for epistemic logic (Halpern and Moses 1992). The crucial difference is that, besides labelling worlds (with the subformulas of φ that are true), the algorithm also labels relations (case $[\mathsf{S}\colon\chi]^\sigma$). With this, it is possible to keep track of which relations 'survive' the model transformations indicated by the partial communication modalities. This labelling of relations is then used when evaluating formulas with the epistemic operators $(\mathrm{D}_{\mathsf{G}}\,\chi)^\sigma$: one only needs to evaluate χ in those worlds accessible via relations that have 'survived' up to the current stage of the run.

Algorithm 1 An algorithm for model checking for $\mathcal{L}_{\mathsf{PC}}$

procedure GlobalMC(M, φ)
  for all $\psi^\sigma \in \text{subm}(\varphi)$ **do**
    for all $w \in W$ **do**
      case $\psi^\sigma = p^\sigma$
        if $w \in V(p)$ **then**
          label w with p^σ
      case $\psi^\sigma = (\neg\chi)^\sigma$
        if w is not labelled with χ^σ **then**
          label w with $(\neg\chi)^\sigma$
      case $\psi^\sigma = (\chi \wedge \xi)^\sigma$
        if w is labelled with χ^σ and ξ^σ **then**
          label w with $(\chi \wedge \xi)^\sigma$
      case $\psi^\sigma = (\mathrm{D}_{\mathsf{G}}\,\chi)^\sigma$
        check ← *true*
        for all $(w, v) \in R_{\mathsf{G}}$ **do**
          if (w, v) is labelled with σ **then**
            if v is not labelled with χ^σ **then**
              check ← *false*
              break
        if *check* **then**
          label w with $(\mathrm{D}_{\mathsf{G}}\,\chi)^\sigma$
      case $\psi^\sigma = [\mathsf{S}\colon\chi]^\sigma$
        for all $\mathtt{i} \in \mathtt{A}$ **do**
          for all $(v, u) \in R_{\mathtt{i}}$ **do**
            if (v, u) is labelled with σ **then**
              if v is labelled with χ iff u is labelled with χ **then**
                label (v, u) with $\sigma, [\mathsf{S}\colon\chi]$
              else

check ← *true*
for all $\mathtt{j} \in \mathtt{S}$ **do**
  if $(v, u) \notin R_{\mathtt{j}}$ **then**
    check ← *false*
    break
if *check* **then**
  label (v, u) with $\sigma, [\mathtt{S}\colon \chi]$
case $\psi^{\sigma} = ([\mathtt{S}\colon \chi]\, \xi)^{\sigma}$
  if w is labelled with $\xi^{\sigma \cdot [\mathtt{S}\colon \chi]}$ **then**
    label w with $([\mathtt{S}\colon \chi]\, \xi)^{\sigma}$

Correctness of the algorithm can be shown by an induction on φ, noting that cases of the algorithm mimic the definition of semantics. From a computational perspective, the preparation of the ordered list from $\mathrm{subm}(\varphi)$ can be done in $O(|\varphi|^2)$ steps: one loop to go over the elements of $\mathrm{subm}(\varphi)$, and a nested loop to compare the current element ψ_1^{σ} to other elements ψ_2^{τ} according to the introduced ordering procedure. The running time of GlobalMC is bounded by $O(|\varphi| \cdot |W| \cdot |\mathtt{A}|^2 \cdot |R|)$ for the case of $[\mathtt{S}\colon \chi]^{\sigma}$.

Theorem 5 *The model checking problem for $\mathcal{L}_{\mathsf{PC}}$ is in P.* ■

3.2 Partial communication vs. public announcements

The action of partial communication is, in a sense, related to that of a public announcement: both are epistemic actions through which agents receive information about the truth-value of a specific formula. Still, there is an important difference: while in a public announcement the information comes from an external source, in partial communication the information comes from agents in the model. It makes sense to discuss the relationship between their formal representations.

Under its standard definition (Plaza 1989), the public announcement of a formula ξ transforms a model by eliminating all $\neg\xi$-worlds. For a fair comparison with the partial communication action, here is an alternative public announcement definition that rather removes edges connecting worlds that disagree on ξ's truth-value (van Benthem and Liu 2007).[10]

Definition 3.4 (Public announcement) Let $M = \langle W, R, V \rangle$ be a model; take a formula ξ. The model $M_{\xi} = \langle W, R^{\xi}, V \rangle$, which is the result of an external source informing all the agents that ξ is the case, is defined such that

$$R^{\xi}{}_{\mathtt{i}} := R_{\mathtt{i}} \cap \sim^{M}_{\xi}. \qquad ◀$$

Note how the indistinguishability relation for a group of agents $\mathtt{G}$ in the new model, denoted as $R^{\xi}{}_{\mathtt{G}}$, is simply $R_{\mathtt{G}} \cap \sim^{M}_{\xi}$. More importantly, in the model that results from this edge-deleting operation, the ξ-region (the partition containing the worlds satisfying ξ) is collectively $\mathtt{P}$-bisimilar (and, in fact, identical) to the model produced by the standard world-removing version. Thus, when evaluating formulas on worlds in this ξ-region, the outcomes from both operations are, as far as $\mathcal{L}$ can tell, the same. That we do stay in this region is

[10] Cf. Gerbrandy and Groeneveld (1997), which removes only edges pointing to $\neg\xi$-worlds. The option used here has the advantage of behaving, with respect to the preservation of certain relational properties, as the standard definition does (see the discussion after Definition 3.4).

guaranteed by the precondition in the semantic interpretation of the modality $[\xi]$ below (Definition 3.5). It is also useful to notice that the operation preserves reflexivity, transitivity and symmetry: if $R_{\mathtt{i}}$ has any of those properties, then so has ${R^{\xi}}_{\mathtt{i}}$, as it is then the intersection of two reflexive, transitive and symmetric relations.

Proposition 1 *Let $M = \langle W, R, V\rangle$ be a model, and let ξ be a formula. Recall (Plaza 1989) that the world-removing public announcement of ξ on M yields the model $M'_{\xi} = \langle [\![\xi]\!]^M, \{R'_{\mathtt{i}} \mid \mathtt{i} \in \mathtt{A}\}, V'\rangle$ with*

$$R'_{\mathtt{i}} := R_{\mathtt{i}} \cap ([\![\xi]\!]^M \times [\![\xi]\!]^M) \qquad \textit{and} \qquad V'(p) := V(p) \cap [\![\xi]\!]^M.$$

Now, take any w in the domain of M'_{ξ} (that is, any $w \in [\![\xi]\!]^M$). Then,

$$(M_{\xi}, w) \rightleftarrows_C (M'_{\xi}, w).$$

Proof. Intuitively, the difference between the world-removing and edge-deleting approaches makes no difference for a collective bisimulation: in both cases, the $\neg\xi$-partition becomes inaccessible from the ξ-partition, where the world w lies. Formally, it is enough to prove that the relation

$$Z := \{(u, u) \in (W \times [\![\xi]\!]^M) \mid u \in [\![\xi]\!]^M\}$$

is a collective bisimulation (between M_{ξ} and M'_{ξ}) containing the pair (w, w). Details can be found in the appendix. ∎

For the language, here is a modality for describing the operation's effect.

Definition 3.5 (Modality $[\xi]$) Define $\mathcal{L}_{\mathsf{PA}}[0] := \mathcal{L}$, where PA stands for *public announcements*. Then, define $\mathcal{L}_{\mathsf{PA}}[i{+}1]$ as the result of extending $\mathcal{L}_{\mathsf{PA}}[i]$ with an additional modality $[\xi]$ for $\xi \in \mathcal{L}_{\mathsf{PA}}[i]$. The language $\mathcal{L}_{\mathsf{PA}}$ is the union of all $\mathcal{L}_{\mathsf{PA}}[n]$ with $n \in \mathbb{N}$, thus essentially extending $\mathcal{L}$ with a modality $[\xi]$ for each formula ξ. The set of atoms for formulas in $\mathcal{L}_{\mathsf{PA}}$ is as in Definition 2.3 with the additional clause $\mathrm{at}([\xi]\,\varphi) := \mathrm{at}(\xi) \cup \mathrm{at}(\varphi)$. For the semantic interpretation,

$$(M, w) \Vdash [\xi]\,\varphi \quad \text{iff}_{def} \quad (M, w) \Vdash \xi \ \text{ implies } \ (M_{\xi}, w) \Vdash \varphi.$$

Define $\langle\xi\rangle\,\varphi := \neg\,[\xi]\,\neg\varphi$. Note how this implies $\Vdash \langle\xi\rangle\,\varphi \leftrightarrow (\xi \wedge [\xi]\,\varphi)$. ◀

Following the strategy used in the proof of Theorem 4, it can be shown that $\mathcal{L}_{\mathsf{PA}}$ is invariant under collective bisimilarity.

Theorem 6 ($\rightleftarrows_C$ implies $\mathcal{L}_{\mathsf{PA}}$-equivalence) *Let (M, w) and (M', w') be two pointed models; take $\mathtt{Q} \subseteq \mathtt{P}$. If $(M, w) \rightleftarrows_C^{\mathtt{Q}} (M', w')$ then, for every $\psi \in \mathcal{L}_{\mathsf{PA}}$ with $\mathrm{at}(\psi) \subseteq \mathtt{Q}$,*

$$(M, w) \Vdash \psi \quad \textit{if and only if} \quad (M', w') \Vdash \psi.$$

Proof. See the appendix. ∎

Finally, an axiom system can be obtained by using the reduction axioms technique, with the crucial axiom being $[\xi]\,\mathrm{D}_{\mathsf{G}}\,\varphi \leftrightarrow (\xi \rightarrow \mathrm{D}_{\mathsf{G}}\,[\xi]\,\varphi)$ (Wáng and Ågotnes 2013). As before, the existence of the reduction axioms implies $\mathcal{L}_{\mathsf{PA}} \preccurlyeq \mathcal{L}$. This, together with the straightforward $\mathcal{L} \preccurlyeq \mathcal{L}_{\mathsf{PA}}$, implies $\mathcal{L} \approx \mathcal{L}_{\mathsf{PA}}$:

the languages $\mathcal{L}$ and $\mathcal{L}_{\mathsf{PA}}$ are equally expressive. With the basics of the edge-deleting public announcement presented/recalled, it is now possible to compare it with the partial communication proposal.

When comparing the partial communication and public announcements settings, a natural question is about the languages' relative expressivity. The answer is simple: $\mathcal{L}_{\mathsf{PC}}$ and $\mathcal{L}_{\mathsf{PA}}$ are both reducible to $\mathcal{L}$, and thus they are equally expressive.

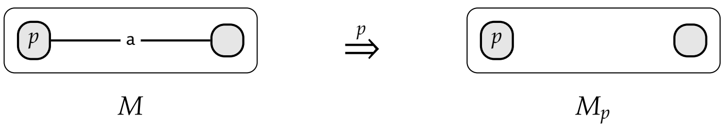
Then, at the semantic level, one might wonder whether the operations can 'mimic' each other. More precisely, one can ask the following.

- Given $\xi \in \mathcal{L}$, are there $\mathtt{S} \subseteq \mathtt{A}$ and $\chi \in \mathcal{L}$ such that $M_{\xi} \rightleftarrows_{C} M_{\mathtt{S}:\chi}$ for every M? In symbols: does $\forall \xi \,.\, \exists \mathtt{S} \,.\, \exists \chi \,.\, \forall M \,.\, (M_{\xi} \rightleftarrows_{C} M_{\mathtt{S}:\chi})$ hold?
- Given $\mathtt{S} \subseteq \mathtt{A}$ and $\chi \in \mathcal{L}$, is there $\xi \in \mathcal{L}$ such that $M_{\mathtt{S}:\chi} \rightleftarrows_{C} M_{\xi}$ for every M? In symbols: does $\forall \mathtt{S} \,.\, \forall \chi \,.\, \exists \xi \,.\, \forall M \,.\, (M_{\mathtt{S}:\chi} \rightleftarrows_{C} M_{\xi})$ hold?

Some known model-update operations have this relationship. For example, the action models of Baltag et al. (1998) generalise standard public announcements: for every formula ξ there is an action model that, when applied to any relational model, produces exactly the one that a public announcement of ξ does. For another example, edge-deleting versions of a public announcement (both that in Gerbrandy and Groeneveld 1997 and that in Definition 3.4, borrowed from van Benthem and Liu 2007) can be represented within the arrow update framework of Kooi and Renne (2011), as it will be discussed later (Subsection 3.3).

Here, the answer to the first question is straightforward: the agents might not have, even together, the information that a public announcement provides.

Fact 1 *Take* $\mathtt{A} = \{\mathsf{a}\}$ *and* $\mathtt{P} = \{p\}$*; consider the (reflexive and symmetric) model* M *below on the left. A public announcement of* p *yields the model* M_p *on the right.*

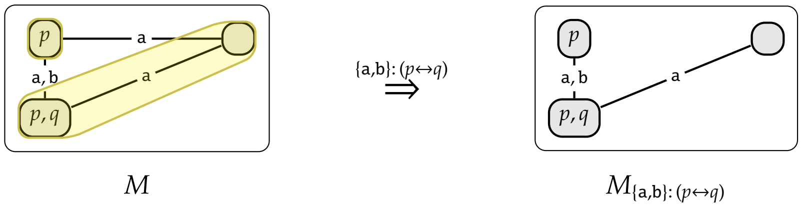


Now, there are no $\mathtt{S} \subseteq \mathtt{A}$ *and* $\chi \in \mathcal{L}$ *such that* $M_{\mathtt{S}:\chi} \rightleftarrows_{C} M_p$*. The group* $\mathtt{S}$ *can be only* $\varnothing$ *or* $\{\mathsf{a}\}$ *and, in both cases,* $R^{\mathtt{S}:\chi}{}_{\mathsf{a}} = R_{\mathsf{a}}$*, regardless of the formula* χ*.* ∎

Thus, $\forall M \,.\, \forall \xi \,.\, \exists \mathtt{S} \,.\, \exists \chi \,.\, (M_{\xi} \rightleftarrows_{C} M_{\mathtt{S}:\chi})$ fails: for the given model, the effect of a public announcement of p cannot be replicated by any act of partial communication. This answers negatively the (stronger) first question above: there are no agents $\mathtt{S}$ and topic χ that can replicate the given public announcement in every model.

The answer to the second question is interesting: through partial communication, the agents can reach epistemic situations that cannot be reached by a public announcement.

Fact 2 *Take* $\mathtt{A} = \{\mathsf{a}, \mathsf{b}\}$ *and* $\mathtt{P} = \{p, q\}$*; consider the (reflexive and symmetric) model* M *below on the left. A partial communication between all agents about* $p \leftrightarrow q$ *(equivalence classes highlighted) yields the model* $M_{\{\mathsf{a},\mathsf{b}\}:(p \leftrightarrow q)}$ *on the right.*



Now, there is no $\xi \in \mathcal{L}$ such that $M_\xi \rightleftarrows_C M_{\{\mathsf{a},\mathsf{b}\}:(p\leftrightarrow q)}$. For this, note that a public announcement preserves the transitivity of indistinguishability relations; yet, while all relations in M are transitive, that for a *in $M_{\{\mathsf{a},\mathsf{b}\}:(p\leftrightarrow q)}$ is not.* ■

Thus, $\forall M\,.\,\forall \mathtt{S}\,.\,\forall \chi\,.\,\exists \xi\,.\,(M_{\mathtt{S}:\chi} \rightleftarrows_C M_\xi)$ fails: for the given model, the effect of a 'conversation' among a and b on $p \leftrightarrow q$ cannot be replicated by any public announcement. This answers negatively the (stronger) second question above: there is no χ that can replicate the given partial communication in every model.

3.3 Partial communication vs. arrow updates

While a public announcement removes *all* edges between worlds disagreeing on the truth-value of the given formula, the *arrow update* framework (Kooi and Renne 2011) allows for a more refined transformation of a model's relations. An arrow update U is a finite set of edge specifications represented by triples of the form $(\xi, \mathtt{i}, \chi)$. Intuitively, each triple in U prescribes to retain, in the updated model, those edges labelled with $\mathtt{i}$ that go from a ξ-world to a χ-world. In this way, arrow updates can target particular edges in a model.

Definition 3.6 (Modality $[U]$) Define $\mathcal{L}_{\mathsf{AU}}[0] := \mathcal{L}$, where AU stands for *arrow updates*. Then, define $\mathcal{L}_{\mathsf{AU}}[i{+}1]$ as the result of extending $\mathcal{L}_{\mathsf{AU}}[i]$ with an additional modality $[U]$, for U a finite list $(\xi_1, \mathtt{i}_1, \chi_1), \ldots, (\xi_m, \mathtt{i}_m, \chi_m)$ with $\xi_j, \chi_j \in \mathcal{L}_{\mathsf{AU}}[i]$ and $\mathtt{i}_j \in \mathtt{A}$ for $1 \leqslant j \leqslant m$. The language $\mathcal{L}_{\mathsf{AU}}$ is the union of all $\mathcal{L}_{\mathsf{AU}}[n]$ with $n \in \mathbb{N}$. The set of atoms for formulas in $\mathcal{L}_{\mathsf{AU}}$ is as in Definition 2.3 plus the clause $\mathsf{at}([U]\,\varphi) := \bigcup_{(\xi,\mathtt{i},\chi)\in U}(\mathsf{at}(\xi) \cup \mathsf{at}(\chi)) \cup \mathsf{at}(\varphi)$. The semantics of arrow update formulas is defined as

$$(M, w) \Vdash [U]\,\varphi \quad \text{iff}_{def} \quad (M_U, w) \Vdash \varphi,$$

where $M_U = \langle W, R^U, V\rangle$ and

$$R^U{}_{\mathtt{i}} := \{(u, u') \in R_{\mathtt{i}} \mid \exists(\xi, \mathtt{i}, \chi) \in U : (M, u) \Vdash \xi \text{ and } (M, u') \Vdash \chi\}.$$ ◀

For structural invariance, that collectively bisimilarity implies equivalence w.r.t. formulas in $\mathcal{L}_{\mathsf{AU}}$ can be shown by a straightforward extension of the corresponding proof (van Ditmarsch et al. 2017, Lemma 3) for the original arrow update language, which lacks the distributed knowledge modality (instead using only knowledge modalities for single agents; Kooi and Renne 2011).

Theorem 7 ($\rightleftarrows_C$ implies $\mathcal{L}_{\mathsf{AU}}$-equivalence) *Let (M, w) and (M', w') be two pointed models; take* $\mathtt{Q} \subseteq \mathtt{P}$. *If $(M, w) \rightleftarrows_C^{\mathtt{Q}} (M', w')$ then, for every $\psi \in \mathcal{L}_{\mathsf{AU}}$ with* $\mathsf{at}(\psi) \subseteq \mathtt{Q}$,

$$(M, w) \Vdash \psi \quad \textit{if and only if} \quad (M', w') \Vdash \psi.$$ ■

For the axiomatisation, we need to resolve a technicality. The original arrow update language lacks the distributed knowledge modality, but the just-defined $\mathcal{L}_{\mathsf{AU}}$ uses it. Thus, we cannot reuse reduction axioms of the original paper 'as is'; some gentle modification is required.

The crucial reduction axiom from Kooi and Renne (2011) is

$$[U]\,\mathrm{K}_{\mathtt{i}}\,\varphi \leftrightarrow \bigwedge_{(\xi,\mathtt{i},\chi)\in U} (\xi \to \mathrm{K}_{\mathtt{i}}(\chi \to [U]\,\varphi)).$$

Now, given an arrow update U and a group of agents $\mathtt{G}$, we can construct sets of triples of the form $\{(\bigwedge_{\mathtt{i}\in\mathtt{G}} \xi_{\mathtt{i}}, \mathtt{i}, \bigwedge_{\mathtt{i}\in\mathtt{G}} \chi_{\mathtt{i}}) \mid \mathtt{i} \in \mathtt{G}\}$, where for each $\mathtt{i} \in \mathtt{G}$ there is one $\xi_{\mathtt{i}}$ (resp. one $\chi_{\mathtt{i}}$) in $\bigwedge_{\mathtt{i}\in\mathtt{G}} \xi_{\mathtt{i}}$ (resp. $\bigwedge_{\mathtt{i}\in\mathtt{G}} \chi_{\mathtt{i}}$) such that $(\xi_{\mathtt{i}}, \mathtt{i}, \chi_{\mathtt{i}}) \in U$. Setting $\xi_{\mathtt{G}} := \bigwedge_{\mathtt{i}\in\mathtt{G}} \xi_{\mathtt{i}}$ and $\chi_{\mathtt{G}} := \bigwedge_{\mathtt{i}\in\mathtt{G}} \chi_{\mathtt{i}}$, the set of all such triples $\{(\xi_{\mathtt{G}}, \mathtt{i}, \chi_{\mathtt{G}}) \mid \mathtt{i} \in \mathtt{G}\}$ is denoted by $U^{\mathtt{G}}$. Intuitively, an edge labelled with $\mathtt{G}$ from u to u' is preserved in a model M if there are triples $\{(\xi_{\mathtt{G}}, \mathtt{i}, \chi_{\mathtt{G}}) \mid \mathtt{i} \in \mathtt{G}\} \subseteq U^{\mathtt{G}}$ such that $(M, u) \Vdash \xi_{\mathtt{G}}$ and $(M, u') \Vdash \chi_{\mathtt{G}}$ for each $\mathtt{i} \in \mathtt{G}$. Hence, the reduction axiom for the interaction of arrow updates and distributed knowledge is

$$[U]\,\mathrm{D}_G\,\varphi \leftrightarrow \bigwedge_{\{(\xi_{\mathtt{G}},\mathtt{i},\chi_{\mathtt{G}})\mid\mathtt{i}\in\mathtt{G}\}\subseteq U^{\mathtt{G}}} (\xi_{\mathtt{G}} \to \mathrm{D}_{\mathtt{G}}(\chi_{\mathtt{G}} \to [U]\,\varphi)).$$

The soundness of the axiom can be shown similarly to the soundness proof from Kooi and Renne (2011). The completeness of the system resulting from adding this axiom can be proved with the standard 'reduction axioms' technique. This shows that $\mathcal{L}$ and $\mathcal{L}_{\mathsf{AU}}$ are equally expressive, which then implies that so are the latter and the partial communication language $\mathcal{L}_{\mathsf{PC}}$.

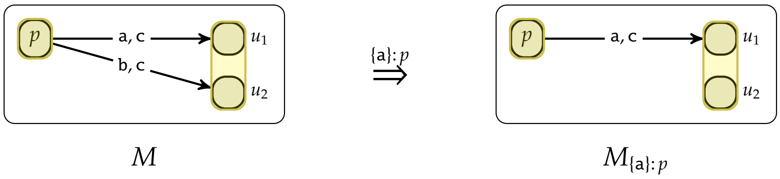
This changes slightly once we compare update expressivity. On the one hand, the effects of certain arrow updates cannot be replicated by partial communication. This follows from Fact 1 and the fact that the effect of an edge-removing public announcement with ξ (Definition 3.4) can be modelled by the arrow update $\{(\xi, \mathtt{i}, \xi), (\neg\xi, \mathtt{i}, \neg\xi) \mid \mathtt{i} \in \mathtt{A}\}$.[11]

Fact 3 *Take $\mathtt{A} = \{\mathtt{a}\}$ and $\mathtt{P} = \{p\}$; consider the (reflexive and symmetric) models M and M_p from Fact 1. When applied to M, the arrow update $U := \{(p, \mathtt{a}, p), (\neg p, \mathtt{a}, \neg p)\}$ will produce the model M_p. Still, as argued in Fact 1, no partial communication can transform M into a model that is collectively bisimilar to M_p.* ∎

On the other hand, partial communication modalities can cut relations to collectively bisimilar states, which cannot be replicated by any arrow updates. Hence, partial communication and arrow updates are, update expressivity wise, incomparable.

Fact 4 *Take $\mathtt{A} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ and $\mathtt{P} = \{p\}$; consider the model M below on the left. A partial communication of agent $\mathtt{a}$ on topic p (equivalence classes highlighted) yields the model $M_{\{\mathtt{a}\}:p}$ on the right.*

[11] As shown in Kooi and Renne (2011), arrow updates can also mimic the version of public announcements from Gerbrandy and Groeneveld (1997), which removes only edges pointing to $\neg\xi$-worlds (via the arrow update $\{(\top, i, \xi) \mid i \in \mathtt{A}\}$) as well as the world removing versions from (Plaza 1989) (via $\{(\top, i, \xi) \mid i \in \mathtt{A}\}$ and an adequate additional modality).



■

However, there is no arrow update U such that M_U is collectively bisimilar *to $M_{\{a\}:p}$. For doing so, one would need a clause (ξ, c, χ) in U with a formula $\chi \in \mathcal{L}_{AU}$ that is, in M, true at u_1 (so the c-edge to u_1 is preserved) but false at u_2 (so the c-edge to u_2 is removed). However, this is impossible: u_1 and u_2 are collectively bisimilar to one another (both are dead-end states), and thus they cannot be distinguished by the language.*

3.4 Discussion

This section has studied further the partial communication framework. Thus, it makes sense to argue for its use, contrasting the choices made with their alternatives.

A first concern might be that, while communication between agents is a crucial form of interaction, it can be already modelled through public announcements (e.g., Ågotnes et al. 2010, van Ditmarsch 2014). Still, this strategy might not be fully suited. In such a setting, an announcement requires, in fact, two parameters: the announcement's precondition and the information the agents receive. When the announcement comes from a 'nameless' external source, it is clear what these two parameters are, and they turn out to be the same: to be 'announced', ξ must be true (the precondition), and the agents learn that ξ is the case (the information).[12] But when the information comes from an agent, precondition and information content are not straightforward, and they might differ. When *an agent* i announces ξ, what is the precondition? There is an announcer involved, so it cannot be only ξ. Is it enough that the announcer knows ξ (i.e., $K_i\, \xi$), or should she be introspective about it (i.e., $K_i\, K_i\, \xi$)? Analogously, what is what the other agents learn? Assuming they trust the announcer, they learn that ξ is true. Do they also learn that the announcer knows ξ (i.e., $K_i\, \xi$), or even that she knows that she knows ξ (i.e., $K_i\, K_i\, \xi$)?

These questions naturally extend to situations of group communication. In group announcement logic (Ågotnes et al. 2010), an announcement from a group S is represented by the public announcement of $\bigwedge_{i \in S} K_i\, \xi_i$, a conjunction specifying a formula ξ_i known by each agent i. In other words, an announcement from a group S is modelled as a parallel action in which each agent $i \in S$ announces a formula she knows. However, other readings may be more appropriate: the group might announce something that is common knowledge among its members, or even announce something they all know distributively. These alternative readings are more naturally represented by the actions introduced in Baltag (2010), Ågotnes and Wáng (2017), Baltag and Smets (2020), of which partial communication is just a (topic-oriented) variation.

Then, in the partial communication setting, although only some of the agents share, this information is received by every agent. This 'everybody hears' set-

[12] More precisely, they learn that ξ was the case at the moment of its announcement.

ting is useful, e.g., for modelling classroom scenarios where (hopefully) everybody listens to what is being told, but only the lecturer and some adventurous students communicate. It can be also used for representing situations similar to public debates, where everybody 'hears' but only the appointed ones get to 'talk'. It can even be used when the communication channel is insecure, and thus privacy cannot be assumed. Of course, it is also interesting to look into more complex 'private communication' scenarios, such as those in which only some agents receive the shared information.[13] Instead, this paper has rather focused on the strategic aspects that arise in competitive situations. In such cases, one naturally wonders whether there is a form of partial communication that can achieve a given goal (e.g., van Ditmarsch 2003). The arbitrary partial communication of Section 4 can help to answer such questions.

4 Arbitrary partial communication

The partial communication framework allows us to model inter-agent information exchange. Yet, consider competitive scenarios. While it is interesting to find out what a form of partial communication can achieve (fix the agents and the topic, then find the consequences), one might be also interested in deciding whether a given goal can be achieved by *some* form of partial communication (fix the *goal*: is there a group of agents and a topic that can achieve it?). This *quantification* over the sharing agents and the topic they discuss adds a strategic dimension to the framework. This is particularly useful when communication occurs over an insecure channel, as one would like to know *whether* there is a form of partial communication (who talks, and on which topic) that can achieve a given goal (e.g., make something common knowledge for a group of agents, while also precluding adversaries/eavesdroppers from learning it, as in van Ditmarsch 2003). Thus, in the spirit of Balbiani et al. (2008), one can then *quantify*, either over the agents that communicate or over the topic they discuss.

Quantifying over the communicating agents does not need additional machinery: $\mathtt{A}$ is finite, so a modality stating that "*φ is true after any group of agents share all their information about χ*" is definable as $[*\colon \chi]\,\varphi := \bigwedge_{\mathtt{S}\subseteq\mathtt{A}}[\mathsf{S}\colon \chi]\,\varphi$ (and thus, by defining $\langle *\colon \chi\rangle\,\varphi := \neg\,[*\colon \chi]\,\neg\varphi$, it follows that $\Vdash \langle *\colon \chi\rangle\,\varphi \leftrightarrow \bigvee_{\mathtt{S}\subseteq\mathtt{A}}\langle \mathsf{S}\colon \chi\rangle\,\varphi$). Hence, in the rest of the section we focus on quantification over topics.

4.1 Syntax, semantics, and model checking

Definition 4.1 (Modality $[\mathsf{S}\colon *]$) Define $\mathcal{L}^*_{\mathsf{PC}}[0]$ as $\mathcal{L}$ plus the *quantifying* modality $[\mathsf{S}\colon *]$. Then, define $\mathcal{L}^*_{\mathsf{PC}}[i{+}1]$ as the result of extending $\mathcal{L}^*_{\mathsf{PC}}[i]$ with an additional modality $[\mathsf{S}\colon \chi]$ for $\mathtt{S} \subseteq \mathtt{A}$ and $\chi \in \mathcal{L}^*_{\mathsf{PC}}[i]$. The language $\mathcal{L}^*_{\mathsf{PC}}$ is the union of all $\mathcal{L}^*_{\mathsf{PC}}[n]$ with $n \in \mathbb{N}$, thus essentially extending $\mathcal{L}_{\mathsf{PC}}$ with a modality $[\mathsf{S}\colon *]$ for each group of agents $\mathtt{S} \subseteq \mathtt{A}$. The set of atoms and size for $\varphi \in \mathcal{L}^*_{\mathsf{PC}}$ extend Definition 3.2 with the clauses $\mathrm{at}([\mathsf{S}\colon *]\,\varphi) := \mathrm{at}(\varphi)$ and $|[\mathsf{S}\colon *]\,\varphi| := |\varphi| + 1$, respectively. For the semantic interpretation,

$$
\begin{aligned}
(M, w) \Vdash [\mathsf{S}\colon *]\,\varphi \quad &\text{iff}_{def} \quad (M, w) \Vdash [\mathsf{S}\colon \chi]\,\varphi \text{ for every } \chi \in \mathcal{L} \\
&\text{iff} \quad (M_{\mathsf{S}\colon \chi}, w) \Vdash \varphi \text{ for every } \chi \in \mathcal{L}.
\end{aligned}
$$

[13]The interested reader is referred, e.g., to the semi-private communication within groups of Ågotnes and Wáng (2017) and the secret 'hacking' from Baltag and Smets (2020).

$\mathsf{A}_{\mathsf{S}:*}$:	$\vdash [\mathsf{S}\colon *]\,\varphi \to [\mathsf{S}\colon \chi]\,\varphi$	for every $\chi \in \mathcal{L}$
$\mathsf{R}_{\mathsf{S}:*}$:	If $\vdash \eta([\mathsf{S}\colon \chi]\,\varphi)$ for all $\chi \in \mathcal{L}$, then $\vdash \eta([\mathsf{S}\colon *]\,\varphi)$	

Table 3: Axiom and rule of inference for the arbitrary case.

If one defines $\langle \mathsf{S}\colon * \rangle\,\varphi := \neg\,[\mathsf{S}\colon *]\,\neg\varphi$, then

$$(M, w) \Vdash \langle \mathsf{S}\colon * \rangle\,\varphi \;\text{ iff}_{def}\; \text{ there is } \chi \in \mathcal{L} \text{ such that } (M_{\mathsf{S}:\chi}, w) \Vdash \varphi. \qquad \blacktriangleleft$$

Note how $[\mathsf{S}\colon *]$ quantifies over formulas in $\mathcal{L}$, and not over formulas in $\mathcal{L}^*_{\mathsf{PC}}$. As in Balbiani et al. (2008), this is to avoid circularity issues. One could have also chosen to quantify over formulas in $\mathcal{L}_{\mathsf{PC}}$, but $\mathcal{L} \approx \mathcal{L}_{\mathsf{PC}}$ (see the paragraph on expressivity on Page 9) so nothing is lost by using $\mathcal{L}$ instead.[14] Note also how, because of the way $\mathcal{L}^*_{\mathsf{PC}}$ is defined ($[\mathsf{S}\colon *]$ is added at the beginning and not at the end), the topic χ of a partial communication formula $[\mathsf{S}\colon \chi]\,\varphi$ might contain arbitrary partial communication modalities (just as publicly announced formulas might contain arbitrary public announcement modalities in Balbiani et al. 2008).

Axiom system. Axiomatising $\mathcal{L}^*_{\mathsf{PC}}$ requires an additional notion.

Definition 4.2 (Necessity Forms) Given a symbol $\sharp \notin P$, the set of *necessity forms* (Goldblatt 1982) is given by

$$\eta(\sharp) ::= \sharp \mid \phi \to \eta(\sharp) \mid \mathsf{D}_{\mathsf{G}}\,\eta(\sharp) \mid [\mathsf{S}\colon \chi]\,\eta(\sharp)$$

with ϕ an $\mathcal{L}^*_{\mathsf{PC}}$-formula, χ an $\mathcal{L}$-formula, and sets of agents $\mathtt{S}, \mathtt{G} \subseteq \mathtt{A}$. In a necessity form $\eta(\sharp)$, replacing $\sharp$ with a $\mathcal{L}^*_{\mathsf{PC}}$-formula ψ produces another $\mathcal{L}^*_{\mathsf{PC}}$-formula, denoted as $\eta(\psi)$. ◀

The (note: *infinitary*) axiom system for $\mathcal{L}^*_{\mathsf{PC}}$, similar to well-known axiomatisations of other logics of quantified epistemic actions (see van Ditmarsch 2023 for an overview), is given by the axioms and rules on Tables 1, 2 and 3. The axiom $\mathsf{A}_{[\mathsf{S}:*]}$ and the rule $\mathsf{R}_{[\mathsf{S}:*]}$ (Table 3) are the crucial ones for the modality for arbitrary partial communication, and their soundness follows from $[\mathsf{S}\colon *]$'s semantic interpretation. The completeness of the whole system can be proved by combining and adapting techniques from Wáng and Ågotnes (2013) (to deal with distributed knowledge) and Balbiani and van Ditmarsch (2015) (to tackle the quantifying modalities).[15]

Theorem 8 *The axioms and rules on Tables 1, 2 and 3 are sound and (together) complete for $\mathcal{L}^*_{\mathsf{PC}}$.*

Proof. See the appendix. ■

[14] Still, for languages with other types of group knowledge, adding a dynamic modality might influence the expressive power. For more on this (in the context of common knowledge and quantified announcements), the reader is referred to Galimullin and Ågotnes (2021) and Ågotnes and Galimullin (2023).

[15] A relatively similar completeness proof, for a system with distributed knowledge and quantification over public announcements, is presented in Ågotnes et al. (2022).

Structural equivalence. The quantifying modality $[\mathtt{S}{:}\,*]$ is also invariant under collective bisimilarity.

Theorem 9 ($\rightleftarrows_C$ implies $\mathcal{L}^*_{\mathsf{PC}}$-equivalence) *Let (M, w) and (M', w') be two pointed models; take $\mathtt{Q} \subseteq \mathtt{P}$. If $(M, w) \rightleftarrows^{\mathtt{Q}}_{C} (M', w')$ then, for every $\psi \in \mathcal{L}^*_{\mathsf{PC}}$ with $\mathrm{at}(\psi) \subseteq \mathtt{Q}$,*

$$(M, w) \Vdash \psi \quad \textit{if and only if} \quad (M', w') \Vdash \psi.$$

Proof. As the proof of Theorem 4. For details, see the appendix. ∎

Expressivity. Even though the partial communication modality $[\mathtt{S}{:}\,\chi]$ does not increase the expressivity of the basic language $\mathcal{L}$, the modality $[\mathtt{S}{:}\,*]$ makes $\mathcal{L}_{\mathsf{PC}}$ more expressive. The intuition for this is that the quantifying modality allows us to talk about formulas of arbitrary finite modal depth, as well as formulas containing atoms that do not appear explicitly in the formula. Using either of these features, one can derive a contradiction from the assumption that $\mathcal{L}_{\mathsf{PC}}$ and $\mathcal{L}^*_{\mathsf{PC}}$ are equally expressive.

Theorem 10 *The language $\mathcal{L}^*_{\mathsf{PC}}$ is strictly more expressive than $\mathcal{L}_{\mathsf{PC}}$.*

Proof. (Sketch) This result can be proved as the analogous result for arbitrary public announcements (Balbiani et al. 2008, Proposition 3.13) and arbitrary group announcements (Ågotnes et al. 2022). Assume, towards a contradiction, that $\mathcal{L}^*_{\mathsf{PC}}$ and $\mathcal{L}_{\mathsf{PC}}$ are equally expressive. Then, given a formula in $\mathcal{L}^*_{\mathsf{PC}}$, there is a logically equivalent formula in $\mathcal{L}_{\mathsf{PC}}$. Now, this formula in $\mathcal{L}_{\mathsf{PC}}$ has only a finite number of atoms, and thus one can find an atom p that does not appear in it. However, $[\mathtt{S}{:}\,*]$ in $\mathcal{L}^*_{\mathsf{PC}}$ quantifies over any formula, and thus also over formulas including p. With this, one can build two models where this atom p plays a 'distinguishing' role. Then, using induction, it can be shown that the formula in $\mathcal{L}_{\mathsf{PC}}$ (without p) cannot tell the models apart, while the formula in $\mathcal{L}^*_{\mathsf{PC}}$ (where quantification ranges also over formulas with p) can. ∎

Model checking As it is shown below (Theorem 11), the complexity of the model checking problem for $\mathcal{L}^*_{\mathsf{PC}}$ is *PSPACE*-complete. This is in line with the *PSPACE*-completeness of many other logics of quantified information change, as arbitrary public announcements (Balbiani et al. 2008), group announcement logic (Ågotnes et al. 2010), coalition announcement logic (Alechina et al. 2021) and arbitrary arrow update logic (van Ditmarsch et al. 2017). However, the witness algorithm presented below has an interesting twist. Model checking algorithms for the aforementioned logics include a step of computing a bisimulation contraction of a model, after which the work continues on the contracted model. This is not possible here: a model and its *collective* bisimulation contraction are not collectively bisimilar (Roelofsen 2005), so they might differ in some formulas' truth-value. The algorithm below still computes bisimulation contractions, but uses them just to keep track of bisimilar worlds. The computation continues on the original non-contracted model.

For the complexity result, the definition and fact below will be useful.

Definition 4.3 (S-definable restrictions) Let (M, w) be a pointed model; take $\mathtt{S} \subseteq \mathtt{A}$. A model (N, w) is an $\mathtt{S}$-definable restriction of (M, w) if and only if $(N, w) = (M_{\mathtt{S}:\chi}, w)$ for some $\chi \in \mathcal{L}^*_{\mathsf{PC}}$. ◀

Fact 5 *Let (M, w) be a finite pointed model. Then there is a finite number of* S*-definable restrictions of (M, w).* ■

The *PSPACE* complexity of the model checking problem for $\mathcal{L}^*_{\mathsf{PC}}$ relies on an algorithm $MC(M, w, \varphi)$ that returns *true* if and only if $(M, w) \Vdash \varphi$, and returns *false* if and only if $(M, w) \nVdash \varphi$. The main challenge is that modalities $[\mathsf{S}\colon *]$ quantify over an *infinite* number of formulas. However, for any given *finite* model M, there is only a *finite* number of possible S-definable model restrictions (Fact 5). The proof of the fact that the problem is *PSPACE*-hard uses the classic reduction from the satisfiability of QBF, which is known to be *PSPACE*-complete.

Algorithm 2 An algorithm for model checking for $\mathcal{L}^*_{\mathsf{PC}}$

```
procedure MC(M, w, φ)
    case φ = [S: χ] ψ
        return MC(M_{S:χ}, w, ψ)
    case φ = [S: *] ψ
        Compute collective P-bisimulation contraction |M|^C
        for all S-definable restrictions (N, w) of (M, w) do
            if MC(N, w, ψ) returns false then
                return false
        return true
```

Theorem 11 *The model checking for $\mathcal{L}^*_{\mathsf{PC}}$ is PSPACE-complete.*

Proof. (Sketch) Let (M, w) be a pointed model; take $\varphi \in \mathcal{L}^*_{\mathsf{PC}}$. In Algorithm 2, Boolean cases and the case for D_G are as expected, and thus omitted. The case for $[\mathsf{S}\colon *]$ relies on the construction of S-definable restrictions. The basic idea for that is to consider a subset of all possible bipartitions of (M, w), taking care that bisimilar worlds end up in the same partition. This can be done by checking that, for each world, if it is in a partition, then all worlds in the same collective bisimulation equivalence class are also in the same partition. Collective bisimulation equivalence classes can be computed by, e.g., a modification of Kanellakis-Smolka algorithm (Kanellakis and Smolka 1990) that runs in polynomial time and takes into account not only individual relations but also their intersections. Having computed collective bisimulation equivalence classes of (M, w), one can construct an S-definable restriction of the model by taking a bipartition such that if v belongs to one partition, then all $u \in [v]$ also belong to the same partition, with $[v]$ being a collective bisimulation equivalence class. For an argument that the algorithm is in *PSPACE*, as well as that it is *PSPACE*-hard, see the Appendix.

4.2 Arbitrary partial communication vs. arbitrary public announcements

Subsection 3.2 showed that the languages of partial communication ($\mathcal{L}_{\mathsf{PC}}$) and public announcements ($\mathcal{L}_{\mathsf{PA}}$) are equally expressive. As this subsection shows, this changes when quantifying modalities are added (the arbitrary partial communication of this section vs the arbitrary public announcements of Balbiani et al. 2008). First, the definitions for arbitrary public announcements.

Definition 4.4 Define $\mathcal{L}^{*}_{\mathsf{PA}}[0]$ as $\mathcal{L}$ plus the *quantifying* modality $[*]$. Then, define $\mathcal{L}^{*}_{\mathsf{PA}}[i{+}1]$ as the result of extending $\mathcal{L}^{*}_{\mathsf{PA}}[i]$ with an additional modality $[\xi]$ for $\xi \in \mathcal{L}^{*}_{\mathsf{PA}}[i]$. The language $\mathcal{L}^{*}_{\mathsf{PA}}$ is the union of all $\mathcal{L}^{*}_{\mathsf{PA}}[i]$ with $i \in \mathbb{N}$, thus essentially extending $\mathcal{L}_{\mathsf{PA}}$ with a modality $[*]$. The set of atoms for formulas in $\mathcal{L}^{*}_{\mathsf{PA}}$ is as in Definition 3.5 plus the clause $\mathrm{at}([*]\,\varphi) := \mathrm{at}(\varphi)$. For the semantic interpretation,

$$(M, w) \Vdash [*]\,\varphi \quad \text{iff}_{def} \quad (M, w) \Vdash [\xi]\,\varphi \text{ for every } \xi \in \mathcal{L}.$$ [16]

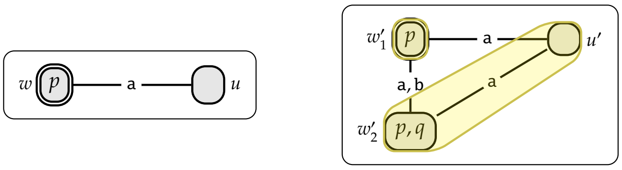
If one defines $\langle * \rangle\,\varphi := \neg\,[*]\,\neg\varphi$, then

$$(M, w) \Vdash \langle * \rangle\,\varphi \quad \text{iff} \quad \text{there is } \xi \in \mathcal{L} \text{ such that } (M, w) \Vdash \langle \xi \rangle\,\varphi.$$ ◀

The theorem below shows that $\mathcal{L}^{*}_{\mathsf{PA}}$ and $\mathcal{L}^{*}_{\mathsf{PC}}$ are incomparable with respect to expressive power (i.e., $\mathcal{L}^{*}_{\mathsf{PC}} \not\preccurlyeq \mathcal{L}^{*}_{\mathsf{PA}}$ and $\mathcal{L}^{*}_{\mathsf{PA}} \not\preccurlyeq \mathcal{L}^{*}_{\mathsf{PC}}$). This result is obtained by adapting techniques and models from Balbiani et al. (2008) and van Ditmarsch et al. (2017) to this partial communication case.

Theorem 12 *$\mathcal{L}^{*}_{\mathsf{PA}}$ and $\mathcal{L}^{*}_{\mathsf{PC}}$ are, expressivity-wise, incomparable.*

Proof. For showing $\mathcal{L}^{*}_{\mathsf{PC}} \not\preccurlyeq \mathcal{L}^{*}_{\mathsf{PA}}$, consider $\langle \{\mathsf{a},\mathsf{b}\} : * \rangle(\mathrm{K}_{\mathsf{b}}\, p \wedge \neg\,\mathrm{K}_{\mathsf{b}}\,\mathrm{K}_{\mathsf{b}}\, p)$ in $\mathcal{L}^{*}_{\mathsf{PC}}$. For a contradiction, assume there is an equivalent $\alpha \in \mathcal{L}^{*}_{\mathsf{PA}}$. Since α is finite there is an atom, say q, that does not occur in it. The strategy consists in building two collectively $\mathtt{P}\backslash\{q\}$-bisimilar pointed models and then argue that, while they can be distinguished by $\langle \{\mathsf{a},\mathsf{b}\} : * \rangle(\mathrm{K}_{\mathsf{a}}\, p \wedge \neg\,\mathrm{K}_{\mathsf{a}}\,\mathrm{K}_{\mathsf{a}}\, p)$, they cannot be distinguished by α. Consider, then, the (reflexive and symmetric) models below for $\mathtt{A} = \{\mathsf{a},\mathsf{b}\}$.



M $\qquad\qquad$ M'

Now, observe the following.

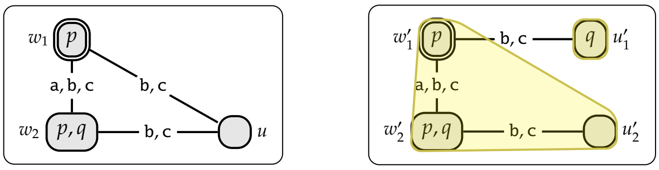
- The formula $\langle \{\mathsf{a},\mathsf{b}\} : * \rangle(\mathrm{K}_{\mathsf{a}}\, p \wedge \neg\,\mathrm{K}_{\mathsf{a}}\,\mathrm{K}_{\mathsf{a}}\, p)$ in $\mathcal{L}^{*}_{\mathsf{PC}}$ can tell (M, w) and (M', w'_1) apart. On the one hand, it fails at (M, w): making $\mathrm{K}_{\mathsf{a}}\, p \wedge \neg\,\mathrm{K}_{\mathsf{a}}\,\mathrm{K}_{\mathsf{a}}\, p$ true at w requires removing the symmetric a-edge between w and u (so $\mathrm{K}_{\mathsf{a}}\, p$ holds), but this makes u inaccessible for a from w (thus $\neg\,\mathrm{K}_{\mathsf{a}}\,\mathrm{K}_{\mathsf{a}}\, p$ fails). On the other hand, it holds at (M', w'_1): a 'conversation' among $\{\mathsf{a},\mathsf{b}\}$ about $p \leftrightarrow q$ produces the desired result (Fact 2).

- The q-less formula α in $\mathcal{L}^{*}_{\mathsf{PA}}$, assumed to be logically equivalent to the distinguishing $\langle \{\mathsf{a},\mathsf{b}\} : * \rangle(\mathrm{K}_{\mathsf{a}}\, p \wedge \neg\,\mathrm{K}_{\mathsf{a}}\,\mathrm{K}_{\mathsf{a}}\, p)$ in $\mathcal{L}^{*}_{\mathsf{PC}}$, cannot tell (M, w) and (M', w'_1) apart. To show this, proceed by structural induction over α. The atomic, Boolean, epistemic and public announcement cases follow from Theorem 6 and the fact that the pointed models are collectively $\mathtt{P}\backslash\{q\}$-bisimilar, witness the relation $\{(w, w'_1), (w, w'_2), (u, u')\}$. For $[*]$ note that, for every announcement in one pointed model, there is a corresponding announcement in the other such that the resulting models remain collectively $\mathtt{P}\backslash\{q\}$-bisimilar. This

[16] Note: $\mathcal{L}^{*}_{\mathsf{PA}}$ extends the language in Balbiani et al. (2008) with distributed knowledge modalities.

is because, in both models, each world is uniquely defined by a Boolean formula containing only atoms p and q. Hence, the aforementioned collective $\mathtt{P}\backslash\{q\}$-bisimulation tells us how to mimic announcements. For example, if a formula ξ with $[\![\xi]\!]^{M'} = \{w'_1, w'_2\}$ is announced on M', one can use the characterising formulas for the collective $\mathtt{P}\backslash\{q\}$-bisimilar w to create, in M, a matching announcement.

For showing $\mathcal{L}^*_{\mathsf{PA}} \not\preccurlyeq \mathcal{L}^*_{\mathsf{PC}}$, proceed in a similar fashion: consider $\langle * \rangle (\mathrm{K_b}\, p \wedge \neg \mathrm{K_b}\, \mathrm{K_b}\, p)$ in $\mathcal{L}^*_{\mathsf{PA}}$ and assume there is an equivalent $\beta \in \mathcal{L}^*_{\mathsf{PC}}$. Let q be an atom not occurring in β, and consider the (reflexive and symmetric) models below for $\mathtt{A} = \{\mathsf{a}, \mathsf{b}, \mathsf{c}\}$.



M $\qquad$ M'

Now, observe the following.

- The formula $\langle * \rangle (\mathrm{K_b}\, p \wedge \neg \mathrm{K_b}\, \mathrm{K_b}\, p)$ in $\mathcal{L}^*_{\mathsf{PA}}$ can tell (M, w_1) and (M', w'_1) apart. On the one hand, it fails at (M, w_1), as an announcement preserves transitivity. On the other hand, it holds at (M', w'_1): the announcement of $q \to p$ (equivalence classes highlighted) produces the desired result.

- The q-less formula β in $\mathcal{L}^*_{\mathsf{PC}}$, assumed to be logically equivalent to the distinguishing $\langle * \rangle (\mathrm{K_b}\, p \wedge \neg \mathrm{K_b}\, \mathrm{K_b}\, p)$ in $\mathcal{L}^*_{\mathsf{PA}}$, cannot tell (M, w_1) and (M', w'_1) apart. To show this, proceed by structural induction over β. The atomic, Boolean, epistemic and partial communication cases follow from Theorem 4 and the fact that the pointed models are collectively $\mathtt{P}\backslash\{q\}$-bisimilar, witness the relation $\{(w_1, w'_1), (w_2, w'_2), (u, u'_1), (u, u'_2)\}$. For $\langle \mathtt{S} : * \rangle$ note that, for every partial communication in one pointed model, there is a corresponding partial communication in the other such that the resulting models remain collectively $\mathtt{P}\backslash\{q\}$-bisimilar. As in the previous case, this is because, in both models, each world is uniquely defined by a Boolean formula containing only atoms p and q. Hence, the aforementioned collective $\mathtt{P}\backslash\{q\}$-bisimulation tells us how to mimic partial communication. For example, if a set of agents $\mathtt{S}$ communicate on M' about a formula χ with $[\![\chi]\!]^{M'} = \{w'_1, u'_1\}$, one can use the characterising formulas for the collective $\mathtt{P}\backslash\{q\}$-bisimilar w_1 and u to create, in M, a matching topic for the same communicating agents. ■

4.3 Arbitrary partial communication vs. arbitrary arrow updates

Subsection 3.3 showed that the languages of partial communication ($\mathcal{L}_{\mathsf{PC}}$) and arrow updates ($\mathcal{L}_{\mathsf{AU}}$) are equally expressive, relying on their reduction to the underlying epistemic logic. Similarly to the previous subsection, allowing quantification over arrow updates (van Ditmarsch et al. 2017) produces a logic that is incomparable to both $\mathcal{L}^*_{\mathsf{PA}}$ (van Ditmarsch et al. 2017, Theorem 1) and $\mathcal{L}^*_{\mathsf{PC}}$ (shown below).

Definition 4.5 Define $\mathcal{L}^*_{\mathsf{AU}}[0]$ as $\mathcal{L}$ plus the *quantifying* modality $[*_U]$. Then, define $\mathcal{L}^*_{\mathsf{AU}}[i{+}1]$ as the result of extending $\mathcal{L}^*_{\mathsf{AU}}[i]$ with an additional modality $[U]$, where U is a finite list $(\xi_1, \mathtt{i}_1, \chi_1), \ldots, (\xi_m, \mathtt{i}_m, \chi_m)$ with $\xi_j, \chi_j \in \mathcal{L}^*_{\mathsf{AU}}[i]$ and $\mathtt{i}_j \in \mathtt{A}$ for $1 \leqslant j \leqslant m$. The language $\mathcal{L}^*_{\mathsf{AU}}$ is the union of all $\mathcal{L}^*_{\mathsf{AU}}[n]$ with $n \in \mathbb{N}$, thus essentially extending $\mathcal{L}_{\mathsf{AU}}$ with a modality $[*_U]$. The set of atoms for formulas in $\mathcal{L}^*_{\mathsf{AU}}$ is as in Definition 3.6 plus the clause $\mathrm{at}([*_U]\,\varphi) := \mathrm{at}(\varphi)$. For the semantic interpretation,

$$(M, w) \Vdash [*_U]\,\varphi \;\text{ iff}_{def}\; (M, w) \Vdash [U]\,\varphi \text{ for every } U \in \mathcal{L}_{\mathsf{AU}}.$$ [17]

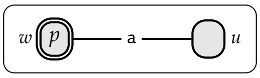
If one defines $\langle *_U \rangle\,\varphi := \neg\,[*_U]\,\neg\varphi$, then

$$(M, w) \Vdash \langle *_U \rangle\,\varphi \;\text{ iff }\; (M, w) \Vdash \langle U \rangle\,\varphi \text{ for some } U \in \mathcal{L}_{\mathsf{AU}}.$$ ◀

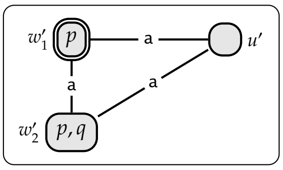
Similarly to Theorem 12, we can show that quantifying over partial communication and quantifying over arrow updates are incomparable to each other. Below we present a proof sketch of this result.

Theorem 13 *$\mathcal{L}^*_{\mathsf{PC}}$ and $\mathcal{L}^*_{\mathsf{U}}$ are, expressivity-wise, incomparable.*

Proof. (Sketch) To see that $\mathcal{L}^*_{\mathsf{AU}} \not\preccurlyeq \mathcal{L}^*_{\mathsf{PC}}$, consider $\langle *_U \rangle(\mathrm{K_a}\, p \wedge \widehat{\mathrm{K}}_\mathsf{a}\, \widehat{\mathrm{K}}_\mathsf{a}\, \neg p)$. For a contradiction, assume there is an equivalent $\alpha \in \mathcal{L}^*_{\mathsf{PC}}$. Pick an atom q not occurring in α; then consider symmetric and reflexive models M and M' below for $\mathtt{A} = \{\mathsf{a}\}$.



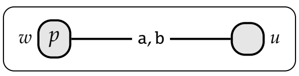
M



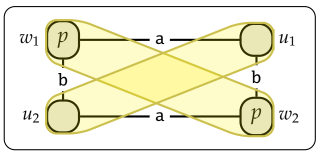
M'

Similarly to the first part of Theorem 12, models are collectively $\mathtt{P}\backslash\{q\}$-bisimilar, and, moreover, agent a, being the single agent in the system, does not have any communication available to her to cut any relations. At the same time, $\langle *_U \rangle(\mathrm{K_a}\, p \wedge \widehat{\mathrm{K}}_\mathsf{a}\, \widehat{\mathrm{K}}_\mathsf{a}\, \neg p)$ does not hold in pointed model (M, w) due to the fact that the first conjunct requires cutting the a-edge from w to u, and the second conjunct requires preserving the very same edge. On the other hand, since each world of M' can be uniquely defined by a Boolean formula (using atom q), we can construct an arrow update U that removes only the arrows between w'_1 and u' and preserves all other arrows. It is then straightforward to verify $(M_U, w'_1) \Vdash \mathrm{K_a}\, p \wedge \widehat{\mathrm{K}}_\mathsf{a}\, \widehat{\mathrm{K}}_\mathsf{a}\, \neg p$, i.e. $(M, w'_1) \Vdash \langle *_U \rangle(\mathrm{K_a}\, p \wedge \widehat{\mathrm{K}}_\mathsf{a}\, \widehat{\mathrm{K}}_\mathsf{a}\, \neg p)$.

For proving $\mathcal{L}^*_{\mathsf{PC}} \not\preccurlyeq \mathcal{L}^*_{\mathsf{AU}}$, consider the models below.



M



M'

[17]Note: $\mathcal{L}^*_{\mathsf{AU}}$ extends the language in van Ditmarsch et al. (2017) with distributed knowledge modalities.

Now consider the formula $\langle \{\mathsf{a},\mathsf{b}\} : * \rangle\, \mathrm{D}_{\{\mathsf{a},\mathsf{b}\}} \perp$ in $\mathcal{L}^*_{\mathsf{PC}}$. Assume that there is an equivalent formula $\alpha \in \mathcal{L}^*_{\mathsf{AU}}$; pick an atom p not occurring in it. It is clear that $(M, w) \nVdash \langle \{\mathsf{a},\mathsf{b}\} : * \rangle\, \mathrm{D}_{\{\mathsf{a},\mathsf{b}\}} \perp$, as none of a and b can tell apart w and u. However, $(M', w_1) \Vdash \langle \{\mathsf{a},\mathsf{b}\} : * \rangle\, \mathrm{D}_{\{\mathsf{a},\mathsf{b}\}} \perp$ (see the highlighted partitions).

To see that α cannot distinguish (M, w) and (M', w_1), first notice that because quantification in $[*_U]$ is implicit, one can use p in the arrow updates we quantify over. Thus, we can force any submodel of (M, w) using $[*_U]$. At the same time, the pairs of worlds (w_1, w_2) and (u_1, u_2) in M' are collectively bisimilar. Thus, we cannot remove an a-edge from the upper part of the model without removing the corresponding edge in the lower part. The same happens with b-edges. This, together with the fact that M and M' are collectively $\mathsf{P}\backslash\{p\}$-bisimilar (a witness is $\{(w, w_1), (w, w_2), (u, u_1), (u, u_2)\}$) implies that $(M, w) \Vdash \alpha$ if and only if $(M', w_1) \Vdash \alpha$. ■

5 Summary and further work

The focus of this paper is the action of *partial communication*. Through it, a group of agents S share, with every agent in the model, all the information they have about the truth-value of a formula χ. Semantically, this is represented by an operation through which the uncertainty of each agent is reduced by removing the uncertainty *about* χ some agent in S has already ruled out. After having recalled the basics of the framework, we showed that its language $\mathcal{L}_{\mathsf{PC}}$ is invariant under collective bisimulation. Moreover, we investigated the complexity of its model checking problem, proving it is in P. It has been also shown that, while the expressivity of $\mathcal{L}_{\mathsf{PC}}$ is exactly that of the languages for public announcements and arrow updates (all are reducible to $\mathcal{L}$), their update expressivity is different. Thus, all three types of communication are incomparable to each other. The focus has then shifted to a modal operator that quantifies over the topic of the communication: a setting for *arbitrary* partial communication. We have provided the operator's semantic interpretation as well as a sound and complete axiom system and invariance results for the resulting language $\mathcal{L}^*_{\mathsf{PC}}$. We have also proved that the model checking problem for the new language $\mathcal{L}^*_{\mathsf{PC}}$ is *PSPACE*-complete, and also showed that $\mathcal{L}^*_{\mathsf{PC}}$ is, expressivity-wise, incomparable to both the language for *arbitrary* public announcements and the language for *arbitrary* arrow updates.

The framework for partial communication provides, arguably, a natural representation of communication between agents. Indeed, it works directly with the information (i.e., uncertainty) the agents have, instead of looking for formulas that are known by the agents, and then using them as announcements (as done, e.g., when dealing with group announcements; Ågotnes et al. 2010). Additionally, the results show that this action is a truly novel epistemic action, different from others as public announcements and arrow updates.

There is further work to do. In the current version of the setting, some questions still demand an answer. An important one is that collective bisimulation is not 'well-behaved': a model and its collective bisimulation contraction are not collectively bisimilar (Roelofsen 2005). One then wonders whether there is a more adequate notion of structural equivalence for the basic language $\mathcal{L}$ and its extensions.

Taking into account that partial communication is incomparable, update expressivity wise, to both public announcements and arrow updates, it may be interesting to determine some special classes of pointed models where all three modalities are equivalent. Moreover, we believe that it is also worthwhile to compare partial communication to other model-changing actions, like, e.g., those of relation-changing logics (Areces et al. 2015).

Alternatively, one can expand the presented framework. For example, one can extend the languages used here by adding a *common knowledge* operator, a step that requires further technical tools (Ågotnes and Wáng 2017, Baltag and Smets 2020, Galimullin and Ågotnes 2021, Ågotnes and Galimullin 2023). Or, one can put further restrictions on communication, like costs and resource bounds (Dolgorukov and Gladyshev 2022, Dolgorukov et al. 2024).

Equally interesting is a generalisation in which the topic of conversation is rather a set of formulas, together with its connection with other forms of communication (e.g., one in which some agents share *all they know* with everybody). Yet another exciting avenue of further research is to consider private or semi-private communication within groups of agents on a given topic.

A Appendix

Proof of Theorem 4

Since $\mathcal{L}_{\mathsf{PC}}$ is the union of $\mathcal{L}_{\mathsf{PC}}[n]$ for all $n \in \mathbb{N}$, the proof will proceed by induction on n. In fact, the manuscript will prove a stronger statement: for every $\psi \in \mathcal{L}_{\mathsf{PC}}$ with $\mathrm{at}(\psi) \subseteq \mathtt{Q}$ and every (M, w) and (M', w'): if $(M, w) \rightleftarrows^{\mathtt{Q}}_{C} (M', w')$ then ***(1)*** $(M, w) \Vdash \psi$ if and only if $(M', w') \Vdash \psi$, and ***(2)*** $(M_{\mathsf{S}:\psi}, w) \rightleftarrows^{\mathtt{Q}}_{C} (M'_{\mathsf{S}:\psi}, w')$. So, take $M = \langle W, R, V \rangle$ and $M' = \langle W', R', V' \rangle$.

Base case. Take $\psi \in \mathcal{L}_{\mathsf{PC}}[0] = \mathcal{L}$ with $\mathrm{at}(\psi) \subseteq \mathtt{Q}$; suppose $(M, w) \rightleftarrows^{\mathtt{Q}}_{C} (M', w')$. In this case, Item ***(1)*** is nothing but Theorem 2. For Item ***(2)***, let Z be the witness for $(M, w) \rightleftarrows^{\mathtt{Q}}_{C} (M', w')$; it will be shown that Z is also a collective $\mathtt{Q}$-bisimulation between $M_{\mathsf{S}:\psi} = \langle W, R^{\mathsf{S}:\psi}, V \rangle$ and $M'_{\mathsf{S}:\psi} = \langle W', R'^{\mathsf{S}:\psi}, V' \rangle$. Take any $(u, u') \in Z$.

- **Atoms**. The operation does not change atomic valuations. Thus, since u and u' agree in all atoms in $\mathtt{Q}$ in M and M' (as Z satisfies **atoms** for those models), they also agree in such atoms in $M_{\mathsf{S}:\psi}$ and $M'_{\mathsf{S}:\psi}$.

- **Forth**. Take any $\mathtt{G} \subseteq \mathtt{A}$ and any $v \in W$ such that $R^{\mathsf{S}:\psi}{}_{\mathtt{G}}uv$. Since $R^{\mathsf{S}:\psi}{}_{\mathtt{G}} = R_{\mathtt{G} \cup \mathtt{S}} \cup (R_{\mathtt{G}} \cap \sim^{M}_{\psi})$ (see observation immediately after Definition 3.1), then $R_{\mathtt{G} \cup \mathtt{S}}uv$ or $(R_{\mathtt{G}} \cap \sim^{M}_{\psi})uv$. ***(i)*** If $R_{\mathtt{G} \cup \mathtt{S}}uv$ then, since Z satisfies **forth** for M and M', there is $v' \in W'$ such that $R'_{\mathtt{G} \cup \mathtt{S}}u'v'$ and $(v, v') \in Z$. Since $R'^{\mathsf{S}:\psi}{}_{\mathtt{G}} = R'_{\mathtt{G} \cup \mathtt{S}} \cup (R'_{\mathtt{G}} \cap \sim^{M'}_{\psi})$, from $R'_{\mathtt{G} \cup \mathtt{S}}u'v'$ it follows that $R'^{\mathsf{S}:\psi}{}_{\mathtt{G}}u'v'$. Thus, this $v' \in W'$ is such that $R'^{\mathsf{S}:\psi}{}_{\mathtt{G}}u'v'$ and $(v, v') \in Z$, as required. ***(ii)*** If $(R_{\mathtt{G}} \cap \sim^{M}_{\psi})uv$, then both $R_{\mathtt{G}}uv$ and $u \sim^{M}_{\psi} v$. From the first and the fact that Z satisfies **forth** for M and M', there is $v' \in W'$ such that $R'_{\mathtt{G}}u'v'$ and $(v, v') \in Z$. Now, $u \sim^{M}_{\psi} v$ indicates that u and v agree on ψ's truth-value. But $\psi \in \mathcal{L}$. Thus, Item ***(1)*** from this base case indicates that u and u' also agree on ψ (as $(u, u') \in Z$), and so do v and v' (from $(v, v') \in Z$). Hence, u' and v' agree on ψ's truth-value, that is,

$u' \sim^{M'}_{\psi} v'$. Therefore, $(R'_{\mathtt{G}} \cap \sim^{M'}_{\psi})uv$, so $R'^{\mathsf{S}:\psi}{}_{\mathtt{G}}u'v'$. This means this $v' \in W'$ is such that $R'^{\mathsf{S}:\psi}{}_{\mathtt{G}}u'v'$ and $(v, v') \in Z$, as required.

- **Back**. As in **forth**, using the fact that Z satisfies **back** for M and M'.

Thus, $M_{\mathsf{S}:\psi} \rightleftarrows^{\mathsf{Q}}_{C} M'_{\mathsf{S}:\psi}$. But $(w, w') \in Z$, so $(M_{\mathsf{S}:\psi}, w) \rightleftarrows^{\mathsf{Q}}_{C} (M'_{\mathsf{S}:\psi}, w')$.

Inductive case. Take $\psi \in \mathcal{L}_{\mathsf{PC}}[i{+}1]$ with $\mathrm{at}(\psi) \subseteq \mathtt{Q}$; suppose $(M, w) \rightleftarrows^{\mathsf{Q}}_{C} (M', w')$. For Item ***(1)***, proceed by structural induction on ψ. The cases for atoms, Boolean operators and D_{G} are as in Theorem 2. The remaining case is for formulas of the form $[\mathsf{S}{:}\,\chi]\varphi$ with $\chi \in \mathcal{L}_{\mathsf{PC}}[i]$, $\varphi \in \mathcal{L}_{\mathsf{PC}}[i{+}1]$ and $\mathrm{at}([\mathsf{S}{:}\,\chi]\varphi) = (\mathrm{at}(\chi) \cup \mathrm{at}(\varphi)) \subseteq \mathtt{Q}$. Here, the structural IH (the one over formulas in $\mathcal{L}_{\mathsf{PC}}[i{+}1]$) states that collectively $\mathtt{Q}$-bisimilar pointed models agree on the truth value of the subformula φ (as $\mathrm{at}(\varphi) \subseteq \mathtt{Q}$). Then, note how, since $\chi \in \mathcal{L}_{\mathsf{PC}}[i]$, $\mathrm{at}(\chi) \subseteq \mathtt{Q}$ and $(M, w) \rightleftarrows^{\mathsf{Q}}_{C} (M', w')$, Item ***(2)*** of the (global) IH implies $(M_{\mathsf{S}:\chi}, w) \rightleftarrows^{\mathsf{Q}}_{C} (M'_{\mathsf{S}:\chi}, w')$. Hence, $(M_{\mathsf{S}:\chi}, w) \Vdash \varphi$ if and only if $(M'_{\mathsf{S}:\chi}, w') \Vdash \varphi$. Now, our case. From left to right, suppose $(M, w) \Vdash [\mathsf{S}{:}\,\chi]\varphi$. By semantic interpretation, $(M_{\mathsf{S}:\chi}, w) \Vdash \varphi$; thus, $(M'_{\mathsf{S}:\chi}, w') \Vdash \varphi$, i.e., $(M', w') \Vdash [\mathsf{S}{:}\,\chi]\varphi$. The right-to-left direction is analogous.

It is only left to prove Item ***(2)*** for $\psi \in \mathcal{L}_{\mathsf{PC}}[i{+}1]$ with $\mathrm{at}(\psi) \subseteq \mathtt{Q}$. This can be done as in the (global) base case, using Item ***(1)*** from this inductive case instead.

Proof of Proposition 1

It will be shown that

$$Z := \{(u, u) \in (W \times [\![\xi]\!]^M) \mid u \in [\![\xi]\!]^M\},$$

is a collective bisimulation. To do so, take any $(u, u) \in Z$ (so $u \in [\![\xi]\!]^M$).

- **Atoms**. Immediate: both operations use the original atomic valuation.

- **Forth**. Take any $\mathtt{G} \subseteq \mathtt{A}$. Suppose there is $v \in W$ such that $R^{\xi}{}_{\mathtt{G}}uv$; it will be shown that v satisfies the requirements. Since $R^{\xi}{}_{\mathtt{G}}uv$, every $\mathtt{i} \in \mathtt{G}$ is such that $R^{\xi}{}_{\mathtt{i}}uv$, that is, $R_{\mathtt{i}}uv$ and $u \sim^{M}_{\xi} v$. The latter and $u \in [\![\xi]\!]^M$ imply $v \in [\![\xi]\!]^M$; thus, $(v, v) \in Z$ and $R'_{\mathtt{i}}uv$. The now latter holds for every $\mathtt{i} \in \mathtt{G}$, which yields the missing piece, $R'_{\mathtt{G}}uv$.

- **Back**. Take any $\mathtt{G} \subseteq \mathtt{A}$. Suppose there is $v \in W$ such that $R'_{\mathtt{G}}uv$; it will be shown that v satisfies the requirements. Since $R'_{\mathtt{G}}uv$, every $\mathtt{i} \in \mathtt{G}$ is such that $R'_{\mathtt{i}}uv$, that is, $R_{\mathtt{i}}uv$ and $\{u, v\} \subseteq [\![\xi]\!]^M$. The latter implies not only $(v, v) \in Z$ but also $u \sim^{M}_{\xi} v$; then, $R^{\xi}{}_{\mathtt{i}}uv$. The now latter holds for every $\mathtt{i} \in \mathtt{G}$, which yields the missing piece, $R^{\xi}{}_{\mathtt{G}}uv$.

For the final detail, note how $(w, w) \in Z$ (as $w \in [\![\xi]\!]^M$).

Proof of Theorem 6

Since $\mathcal{L}_{\mathsf{PA}}$ is the union of $\mathcal{L}_{\mathsf{PA}}[n]$ for all $n \in \mathbb{N}$, the proof will proceed by induction on n. In fact, a stronger statement will be proved: for every $\psi \in \mathcal{L}_{\mathsf{PA}}$ with $\mathrm{at}(\psi) \subseteq \mathtt{Q}$ and every (M, w) and (M', w'), if $(M, w) \rightleftarrows^{\mathsf{Q}}_{C} (M', w')$ then ***(1)*** $(M, w) \Vdash \psi$ if and only if $(M', w') \Vdash \psi$, and ***(2)*** $(M_{\psi}, w) \rightleftarrows^{\mathsf{Q}}_{C} (M'_{\psi}, w')$. Thus, take $M = \langle W, R, V\rangle$ and $M' = \langle W', R', V'\rangle$.

Base case. Take $\psi \in \mathcal{L}_{\mathsf{PA}}[0] = \mathcal{L}$ with $\mathrm{at}(\psi) \subseteq \mathtt{Q}$; suppose $(M, w) \rightleftarrows^{\mathtt{Q}}_{C} (M', w')$. In this case, Item *(1)* is nothing but Theorem 2. For Item *(2)*, let Z be the witness for $(M, w) \rightleftarrows^{\mathtt{Q}}_{C} (M', w')$; it will be shown that Z is also a collective $\mathtt{Q}$-bisimulation between $M_\psi = \langle W, R^\psi, V\rangle$ and $M'_\psi = \langle W', R'^\psi, V'\rangle$. Take any $(u, u') \in Z$.

- **Atoms**. The operation does not change atomic valuations. Thus, since u and u' agree in all atoms in $\mathtt{Q}$ in M and M' (as Z satisfies **atoms** for those models), they also agree in such atoms in M_ψ and M'_ψ.
- **Forth**. Take any $\mathtt{G} \subseteq \mathtt{A}$; suppose there is $v \in W$ such that $R^\psi{}_{\mathtt{G}}uv$. Since $R^\psi{}_{\mathtt{G}} = R_{\mathtt{G}} \cap \sim^M_\psi$, then $R_{\mathtt{G}}uv$ and $u \sim^M_\psi v$. From the former, $(u, u') \in Z$ and Z satisfying **forth** for M and M', there is $v' \in W'$ such that $R'_{\mathtt{G}}u'v'$ and $(v, v') \in Z$. Now, $u \sim^M_\psi v$ says that u and v agree on ψ's truth-value. But $\psi \in \mathcal{L}$. Thus, Item *(1)* from this base case indicates that u and u' also agree on ψ (from $(u, u') \in Z$), and so do v and v' (from $(v, v') \in Z$). Hence, u' and v' agree on ψ's truth-value, that is, $u' \sim^{M'}_\psi v'$. Thus, $R'_{\mathtt{G}}u'v'$ and $u' \sim^{M'}_\psi v'$; hence, $R'^\psi{}_{\mathtt{G}}u'v'$, as actually required.
- **Back**. As in **forth**, using the fact that Z satisfies **back** for M and M'.

Thus, $M_\psi \rightleftarrows^{\mathtt{Q}}_{C} M'_\psi$. Moreover, $(w, w') \in Z$; thus, $(M_\psi, w) \rightleftarrows^{\mathtt{Q}}_{C} (M'_\psi, w')$.

Inductive case. Take $\psi \in \mathcal{L}_{\mathsf{PA}}[i{+}1]$ with $\mathrm{at}(\psi) \subseteq \mathtt{Q}$; suppose $(M, w) \rightleftarrows^{\mathtt{Q}}_{C} (M', w')$. For Item *(1)*, proceed by structural induction on ψ. The cases for atoms, Boolean operators and $\mathrm{D}_{\mathtt{G}}$ are as in Theorem 2. The remaining case is for formulas of the form $[\xi]\varphi$ with $\xi \in \mathcal{L}_{\mathsf{PA}}[i]$, $\varphi \in \mathcal{L}_{\mathsf{PA}}[i{+}1]$ and $\mathrm{at}([\xi]\varphi) = (\mathrm{at}(\xi) \cup \mathrm{at}(\varphi)) \subseteq \mathtt{Q}$. Now, note how $\xi \in \mathcal{L}_{\mathsf{PA}}[i]$ and $(M, w) \rightleftarrows^{\mathtt{Q}}_{C} (M', w')$ imply two facts. First, together with Item *(1)* of the (global) IH, they imply $(M, w) \Vdash \xi$ if and only if $(M', w') \Vdash \xi$. Second, together with Item *(2)* of the same, they yield $(M_\xi, w) \rightleftarrows^{\mathtt{Q}}_{C} (M'_\xi, w')$, which together with the structural IH (collectively $\mathtt{Q}$-bisimilar pointed models agree on the truth value of subformulas of $[\xi]\varphi$ containing only atoms in $\mathtt{Q}$), imply $(M_\xi, w) \Vdash \varphi$ if and only if $(M'_\xi, w') \Vdash \varphi$ (as $\mathrm{at}(\varphi) \subseteq \mathtt{Q}$). Now, our case. From left to right, suppose $(M, w) \Vdash [\xi]\varphi$. By semantic interpretation, $(M, w) \Vdash \xi$ implies $(M_\xi, w) \Vdash \varphi$; thus, $(M', w') \Vdash \xi$ implies $(M'_\xi, w') \Vdash \varphi$, i.e., $(M', w') \Vdash [\xi]\varphi$. The right-to-left direction is analogous.

It is only left to prove Item *(2)* for $\psi \in \mathcal{L}_{\mathsf{PA}}[i{+}1]$ with $\mathrm{at}(\psi) \subseteq \mathtt{Q}$. This can be done as in the (global) base case, using Item *(1)* from this inductive case instead.

Proof of Theorem 8

Recall the additional axiom and rule,

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$$\begin{array}{ll} \mathsf{A}_{\mathsf{S}:*}: & \vdash [\mathsf{S}{:}*]\varphi \rightarrow [\mathsf{S}{:}\chi]\varphi \quad \text{for every } \chi \in \mathcal{L}, \\ \mathsf{R}_{\mathsf{S}:*}: & \text{If} \vdash \eta([\mathsf{S}{:}\chi]\varphi) \text{ for all } \chi \in \mathcal{L}, \text{ then} \vdash \eta([\mathsf{S}{:}*]\varphi), \end{array}$$

as well as the syntax for necessity forms,

$$\eta(\sharp) ::= \sharp \mid \phi \rightarrow \eta(\sharp) \mid \mathrm{D}_{\mathtt{G}}\,\eta(\sharp) \mid [\mathsf{S}{:}\chi]\,\eta(\sharp).$$

Soundness

The soundness of axioms and rules on Tables 1 and 2 has been already established (Theorem 1 and Theorem 3, respectively). For those in Table 3, the soundness of $\mathsf{A}_{\mathsf{S}:*}$ follows directly from the semantic interpretation of $[\mathsf{S}\colon *]$. For $\mathsf{R}_{\mathsf{S}:*}$, note first that the rule is truth-preserving. The proof of this fact relies on the semantics of $[\mathsf{S}\colon *]$, proceeding in this case by induction over necessity forms. Take any pointed model (M, w).

- **Base case ($\eta(\sharp) = \sharp$).** Suppose $(M, w) \Vdash [\mathsf{S}\colon \chi]\,\varphi$ holds for all $\chi \in \mathcal{L}$. Then, the semantics of $[\mathsf{S}\colon *]$ imply $(M, w) \Vdash [\mathsf{S}\colon *]\,\varphi$.

- **Inductive case ($\eta(\sharp) = \phi \to \eta(\sharp)$ with $\phi \in \mathcal{L}^*_{\mathsf{PC}}$).** Suppose $(M, w) \Vdash \phi \to \eta([\mathsf{S}\colon \chi]\,\varphi)$ holds for all $\chi \in \mathcal{L}$; suppose further that $(M, w) \Vdash \phi$. Then, $(M, w) \Vdash \eta([\mathsf{S}\colon \chi]\,\varphi)$ holds for all $\chi \in \mathcal{L}$ and hence, by IH, $(M, w) \Vdash \eta([\mathsf{S}\colon *]\,\varphi)$. Thus, $(M, w) \Vdash \phi \to \eta([\mathsf{S}\colon *]\,\varphi)$.

- **Inductive case ($\eta(\sharp) = \mathrm{D}_{\mathsf{G}}\,\eta(\sharp)$).** Suppose $(M, w) \Vdash \mathrm{D}_{\mathsf{G}}\,\eta([\mathsf{S}\colon \chi]\,\varphi)$ holds for all $\chi \in \mathcal{L}$. By semantic interpretation, for all $u \in W$, if $R_{\mathsf{G}}wu$ then $(M, u) \Vdash \eta([\mathsf{S}\colon \chi]\,\varphi)$, for all $\chi \in \mathcal{L}$. Then, by IH, each such u is such that $(M, u) \Vdash \eta([\mathsf{S}\colon *]\,\varphi)$; thus, $(M, w) \Vdash \mathrm{D}_{\mathsf{G}}\,\eta([\mathsf{S}\colon *]\,\varphi)$.

- **Inductive case ($\eta(\sharp) = [\mathsf{S}\colon \chi]\,\eta(\sharp)$).** Suppose $(M, w) \Vdash [\mathsf{S}'\colon \chi']\,\eta([\mathsf{S}\colon \chi]\,\varphi)$ holds for all $\chi \in \mathcal{L}$. By semantic interpretation, $(M_{\mathsf{S}'\colon \chi'}, w) \Vdash \eta([\mathsf{S}\colon \chi]\,\varphi)$ holds for all $\chi \in \mathcal{L}$. Then, by IH, $(M_{\mathsf{S}'\colon \chi'}, w) \Vdash \eta([\mathsf{S}\colon *]\,\varphi)$ and therefore $(M, w) \Vdash [\mathsf{S}'\colon \chi']\,\eta([\mathsf{S}\colon *]\,\varphi)$.

Since the rule is truth-preserving, it is also validity preserving, which completes the proof.

Completeness

For completeness, the following complexity ordering will be useful.

Definition A.1 The *Boolean, dynamic and quantifier depths* of formulas in $\mathcal{L}^*_{\mathsf{PC}}$ measure, respectively, the number of nested Boolean operators, communication operators and quantifiers. They are given, respectively, by the functions δ_{B}, $\delta_{[]}$ and $\delta_{\forall}$, defined recursively as

$$
\begin{array}{rlrlrl}
\delta_{\mathrm{B}}(p) & := 1 & \delta_{[]}(p) & := 0 & \delta_{\forall}(p) & := 0 \\
\delta_{\mathrm{B}}(\neg\varphi) & := \delta_{\mathrm{B}}(\varphi) + 1 & \delta_{[]}(\neg\varphi) & := \delta_{[]}(\varphi) & \delta_{\forall}(\neg\varphi) & := \delta_{\forall}(\varphi) \\
\delta_{\mathrm{B}}(\varphi \wedge \psi) & := \max(\delta_{\mathrm{B}}(\varphi), \delta_{\mathrm{B}}(\psi)) & \delta_{[]}(\varphi \wedge \psi) & := \max(\delta_{[]}(\varphi), \delta_{[]}(\psi)) & \delta_{\forall}(\varphi \wedge \psi) & := \max(\delta_{\forall}(\varphi), \delta_{\forall}(\psi)) \\
\delta_{\mathrm{B}}(\mathrm{D}_{\mathsf{G}}\,\varphi) & := \delta_{\mathrm{B}}(\varphi) + 1 & \delta_{[]}(\mathrm{D}_{\mathsf{G}}\,\varphi) & := \delta_{[]}(\varphi) & \delta_{\forall}(\mathrm{D}_{\mathsf{G}}\,\varphi) & := \delta_{\forall}(\varphi) \\
\delta_{\mathrm{B}}([\mathsf{S}\colon \chi]\,\varphi) & := \big(8 + \delta_{\mathrm{B}}(\chi)\big)\delta_{\mathrm{B}}(\varphi) & \delta_{[]}([\mathsf{S}\colon \chi]\,\varphi) & := \delta_{[]}(\chi) + \delta_{[]}(\varphi) + 1 & \delta_{\forall}([\mathsf{S}\colon \chi]\,\varphi) & := \delta_{\forall}(\chi) + \delta_{\forall}(\varphi) \\
\delta_{\mathrm{B}}([\mathsf{S}\colon *]\,\varphi) & := \delta_{\mathrm{B}}(\varphi) & \delta_{[]}([\mathsf{S}\colon *]\,\varphi) & := \delta_{[]}(\varphi) & \delta_{\forall}([\mathsf{S}\colon *]\,\varphi) & := \delta_{\forall}(\varphi) + 1
\end{array}
$$

Then, use "⅋" for a natural-language disjunction (just as "&" stands for a natural-language conjunction). The complexity ordering $\prec$ between two formulas φ, ψ in $\mathcal{L}^*_{\mathsf{PC}}$ gives priority to the quantifier depth, then to the dynamic depth and finally to the Boolean depth:

$$
\varphi \prec \psi \quad \text{iff}_{def} \quad ⅋ \left\{
\begin{array}{l}
\delta_{\forall}(\varphi) < \delta_{\forall}(\psi), \\
\delta_{\forall}(\varphi) = \delta_{\forall}(\psi) \;\;\&\;\; \delta_{[]}(\varphi) < \delta_{[]}(\psi), \\
\delta_{\forall}(\varphi) = \delta_{\forall}(\psi) \;\;\&\;\; \delta_{[]}(\varphi) = \delta_{[]}(\psi) \;\;\&\;\; \delta_{\mathrm{B}}(\varphi) < \delta_{\mathrm{B}}(\psi)
\end{array}
\right\} \quad \blacktriangleleft
$$

The main ideas of this completeness proof come from the completeness proofs for APAL (Balbiani and van Ditmarsch 2015) and epistemic logic with distributed knowledge (Fagin et al. 1992). These ideas will be adapted and combined to prove the completeness of the stated proof system for $\mathcal{L}^*_{\mathsf{PC}}$.

It is well-known that the intersection of relations is not modally definable; hence one cannot build a canonical model straight away. The strategy is, instead, to build an intermediate "pseudo-model", where accessibility relations labelled with G are taken as primitive. This pseudo-model can be then unwind into a tree-like model, and then one can show that these structures are collectively bisimilar.

Definition A.2 A *pseudo-model* is a tuple $M = \langle W, \mathcal{R}, V\rangle$ where W and V are as in a model (Definition 2.1), and $\mathcal{R} = \{\mathcal{R}_{\mathtt{i}} \subseteq W \times W \mid \mathtt{i} \in \mathtt{A}\} \cup \{\mathcal{R}_{\mathtt{G}} \subseteq W \times W \mid \mathtt{G} \subseteq \mathtt{A}\}$ assigns a binary relation on W to each agent $\mathtt{i} \in \mathtt{A}$ *and also to every* group *of agents* $\mathtt{G} \subseteq \mathtt{A}$. Moreover, these relations are required to satisfy the following.

(i) $\mathcal{R}_{\{\mathtt{i}\}} = \mathcal{R}_{\mathtt{i}}$, and

(ii) for all $\mathtt{H}, \mathtt{G} \subseteq \mathtt{A}$, if $\mathtt{H} \subseteq \mathtt{G}$ then $\mathcal{R}_{\mathtt{G}} \subseteq \mathcal{R}_{\mathtt{H}}$. ◀

The first difference between a model and a pseudo-model is that, while the first only requires relations $R_{\mathtt{i}}$ for each $\mathtt{i} \in \mathtt{A}$ (building the relations $R_{\mathtt{G}}$ for $\mathtt{G} \subseteq \mathtt{A}$ using intersections), the second requires, additionally, relations for each $\mathtt{G} \subseteq \mathtt{A}$. More importantly, even though the requirements in a pseudo-model guarantee that $\mathcal{R}_{\mathtt{G}} \subseteq \bigcap_{\mathtt{i} \in \mathtt{G}} \mathcal{R}_{\mathtt{i}}$, the subset relation in the other direction might not hold: one can have pairs that are in $\mathcal{R}_{\mathtt{i}}$ for every $\mathtt{i} \in \mathtt{G}$ without being in $\mathcal{R}_{\mathtt{G}}$. This is the main difference w.r.t. models where, by definition, $\bigcap_{\mathtt{i} \in \mathtt{G}} R_{\mathtt{i}} = R_{\mathtt{G}}$. In fact, while every model is a pseudo-model, not every pseudo-model is a model. Still, note how formulas in $\mathcal{L}^*_{\mathsf{PC}}$ can be interpreted in pseudo-models in the same way they are semantically interpreted in models.

While the construction of the canonical model usually requires maximal consistent sets of formulas, the strategy here uses the somewhat different maximal consistent *theories*. Recall that the derivation system under discussion consists of the axioms and rules on Tables 1, 2 and 3.

Definition A.3 Let $\mathcal{APC}$ be the minimal set that contains all the instances of the derivation system's axiom schemata and is closed under all its rules. A set $x \subseteq \mathcal{L}^*_{\mathsf{PC}}$ is called a *theory* if and only if ***(T1)*** $\mathcal{APC} \subseteq x$, ***(T2)*** x is closed under MP (Table 1) and ***(T3)*** x is closed under $\mathsf{R}_{\mathsf{S}:*}$ (Table 3).

A theory x is *consistent* if and only if there is no $\varphi \in \mathcal{L}^*_{\mathsf{PC}}$ such that $\varphi \in x$ and $\neg\varphi \in x$. A theory x is *maximal* if and only if either $\varphi \in x$ or $\neg\varphi \in x$ for all $\varphi \in \mathcal{L}^*_{\mathsf{PC}}$. The smallest theory is $\mathcal{APC}$, and the largest theory is $\mathcal{L}^*_{\mathsf{PC}}$. ◀

Note: while theories are required to be closed under MP and $\mathsf{R}_{\mathsf{S}:*}$, they are not required to be closed under the two other rules of the system, G_{D} and $\mathsf{RE}_{\mathsf{S}:\chi}$. This is because, while MP and $\mathsf{R}_{\mathsf{S}:*}$ preserve both validity and truth, G_{D} and $\mathsf{RE}_{\mathsf{S}:\chi}$ rules preserve validity but not truth.

The following theories will be of great help during the proof.

Lemma 1 *Let x be a theory; take $\varphi, \chi \in \mathcal{L}^*_{\mathsf{PC}}$. Then, all of the following are theories.*

(i) $\varphi \to x := \{\xi \mid \varphi \to \xi \in x\}$,

(ii) $\mathrm{D}_{\mathsf{G}}\, x := \{\xi \mid \mathrm{D}_{\mathsf{G}}\, \xi \in x\}$,

(iii) $[\mathsf{S}\colon \chi]\, x := \{\xi \mid [\mathsf{S}\colon \chi]\, \xi \in x\}$

Proof.

(i) ***(T1)*** Take any $\xi \in \mathcal{APC}$. By propositional reasoning, $\phi \to \xi \in \mathcal{APC}$ for any $\phi \in \mathcal{L}^*_{\mathsf{PC}}$; in particular, $\varphi \to \xi \in \mathcal{APC}$. Since x is a theory, $\mathcal{APC} \subseteq x$, so $\varphi \to \xi \in x$. Hence, by definition, $\xi \in \varphi \to x$.

(T2) Take $\phi \to \xi$ and ϕ in $\varphi \to x$; then, $\varphi \to (\phi \to \xi)$ and $\varphi \to \phi$ are in x. Being a propositional validity, $\big(\varphi \to (\phi \to \xi)\big) \to \big((\varphi \to \phi) \to (\varphi \to \xi)\big)$ is in $\mathcal{APC}$, and thus it is also in x (as x is a theory). But, being a theory, x is closed under MP, so $(\varphi \to \phi) \to (\varphi \to \xi)$ is in x, and thus so is $\varphi \to \xi$. Hence, by definition, $\xi \in \varphi \to x$.

(T3) Suppose $\eta([\mathsf{S}\colon \chi]\, \psi) \in \varphi \to x$ for every $\chi \in \mathcal{L}$; then, $\varphi \to \eta([\mathsf{S}\colon \chi]\, \psi) \in x$ for every $\chi \in \mathcal{L}$. Being a theory, x is closed under $\mathsf{R}_{\mathsf{S}:*}$; moreover, $\varphi \to \eta(\sharp)$ is a necessity form. Hence, $\varphi \to \eta([\mathsf{S}\colon *]\, \psi) \in x$ and thus, by definition, $\eta([\mathsf{S}\colon *]\, \psi) \in \varphi \to x$.

(ii) ***(T1)*** Take any $\xi \in \mathcal{APC}$. From G_{D} it follows that $\mathrm{D}_{\mathsf{G}}\, \xi \in \mathcal{APC}$; but x is a theory, so $\mathcal{APC} \subseteq x$ and hence $\mathrm{D}_{\mathsf{G}}\, \xi \in x$. Thus, by definition, $\xi \in \mathrm{D}_{\mathsf{G}}\, x$.

(T2) Take $\phi \to \xi$ and ϕ in $\mathrm{D}_{\mathsf{G}}\, x$; then, $\mathrm{D}_{\mathsf{G}}(\phi \to \xi)$ and $\mathrm{D}_{\mathsf{G}}\, \phi$ are in x. From axiom K_{D} we have $\mathrm{D}_{\mathsf{G}}(\phi \to \xi) \to (\mathrm{D}_{\mathsf{G}}\, \phi \to \mathrm{D}_{\mathsf{G}}\, \xi) \in \mathcal{APC}$ and thus $\mathrm{D}_{\mathsf{G}}(\phi \to \xi) \to (\mathrm{D}_{\mathsf{G}}\, \phi \to \mathrm{D}_{\mathsf{G}}\, \xi) \in x$. But, being a theory, x is closed under MP, so $\mathrm{D}_{\mathsf{G}}\, \phi \to \mathrm{D}_{\mathsf{G}}\, \xi \in x$ and then $\mathrm{D}_{\mathsf{G}}\, \xi \in x$. Hence, by definition $\xi \in \mathrm{D}_{\mathsf{G}}\, x$.

(T3) Suppose $\eta([\mathsf{S}\colon \chi]\, \varphi) \in \mathrm{D}_{\mathsf{G}}\, x$ for every $\chi \in \mathcal{L}$; then, $\mathrm{D}_{\mathsf{G}}\, \eta([\mathsf{S}\colon \chi]\, \varphi) \in x$ for every $\chi \in \mathcal{L}$. Being a theory, x is closed under $\mathsf{R}_{\mathsf{S}:*}$; moreover, $\mathrm{D}_{\mathsf{G}}\, \eta(\sharp)$ is a necessity form. Hence, $\mathrm{D}_{\mathsf{G}}\, \eta([\mathsf{S}\colon *]\, \varphi) \in x$ and thus, by definition, $\eta([\mathsf{S}\colon *]\, \varphi) \in \mathrm{D}_{\mathsf{G}}\, x$.

(iii) ***(T1)*** Take any $\xi \in \mathcal{APC}$. The rule

$$\text{if} \vdash \phi \text{ then} \vdash [\mathsf{S}\colon \chi]\, \phi$$

is derivable in the system for any S and χ,[18] so $[\mathsf{S}\colon \chi]\, \xi \in \mathcal{APC}$. But x is a theory, so $\mathcal{APC} \subseteq x$ and hence $[\mathsf{S}\colon \chi]\, \xi \in x$. Thus, by definition, $\xi \in [\mathsf{S}\colon \chi]\, x$.

(T2) Take $\phi \to \xi$ and ϕ in $[\mathsf{S}\colon \chi]\, x$. Then, both $[\mathsf{S}\colon \chi](\phi \to \xi)$ and $[\mathsf{S}\colon \chi]\, \phi$ are in x. The axiom

$$\vdash [\mathsf{S}\colon \chi](\phi \to \xi) \to ([\mathsf{S}\colon \chi]\, \phi \to [\mathsf{S}\colon \chi]\, \xi)$$

[18] Suppose $\vdash \phi$. From $\vdash \neg(\neg p \wedge p)$, propositional reasoning yields $\vdash \phi \leftrightarrow \neg(\neg p \wedge p)$ for any atom p. Then, $\mathsf{RE}_{\mathsf{S}:\chi}$ produces the first piece, $\vdash [\mathsf{S}\colon \chi]\, \phi \leftrightarrow [\mathsf{S}\colon \chi]\, \neg(\neg p \wedge p)$. For the second piece, axiom $\mathsf{A}^{\neg}_{\mathsf{S}:\chi}$ yields $\vdash [\mathsf{S}\colon \chi]\, \neg(\neg p \wedge p) \leftrightarrow \neg\, [\mathsf{S}\colon \chi](\neg p \wedge p)$ and axiom $\mathsf{A}^{\wedge}_{\mathsf{S}:\chi}$ yields $\vdash [\mathsf{S}\colon \chi](\neg p \wedge p) \leftrightarrow ([\mathsf{S}\colon \chi]\, \neg p \wedge [\mathsf{S}\colon \chi]\, p)$, so $\vdash \neg\, [\mathsf{S}\colon \chi](\neg p \wedge p) \leftrightarrow \neg([\mathsf{S}\colon \chi]\, \neg p \wedge [\mathsf{S}\colon \chi]\, p)$. From those two, propositional reasoning yields $\vdash [\mathsf{S}\colon \chi]\, \neg(\neg p \wedge p) \leftrightarrow \neg([\mathsf{S}\colon \chi]\, \neg p \wedge [\mathsf{S}\colon \chi]\, p)$. For the third piece, axiom $\mathsf{A}^{\neg}_{\mathsf{S}:\chi}$ yields $\vdash [\mathsf{S}\colon \chi]\, \neg p \leftrightarrow \neg\, [\mathsf{S}\colon \chi]\, p$. Then, propositional reasoning produces $\vdash ([\mathsf{S}\colon \chi]\, \neg p \wedge [\mathsf{S}\colon \chi]\, p) \leftrightarrow (\neg\, [\mathsf{S}\colon \chi]\, p \wedge [\mathsf{S}\colon \chi]\, p)$, from which $\vdash \neg([\mathsf{S}\colon \chi]\, \neg p \wedge [\mathsf{S}\colon \chi]\, p) \leftrightarrow \neg(\neg\, [\mathsf{S}\colon \chi]\, p \wedge [\mathsf{S}\colon \chi]\, p)$ follows. From the three pieces and propositional reasoning, one gets $\vdash [\mathsf{S}\colon \chi]\, \phi \leftrightarrow \neg(\neg\, [\mathsf{S}\colon \chi]\, p \wedge [\mathsf{S}\colon \chi]\, p)$. But the right-hand side of this equivalence is a tautology. Then, by propositional reasoning, $\vdash [\mathsf{S}\colon \chi]\, \phi$.

is derivable in the system,[19] so it is in $\mathcal{APC}$ and thus also in x. But, being a theory, x is closed under MP, so $[\mathsf{S}{:}\,\chi]\,\phi \to [\mathsf{S}{:}\,\chi]\,\xi \in x$ and then $[\mathsf{S}{:}\,\chi]\,\xi \in x$. Hence, by definition $\xi \in [\mathsf{S}{:}\,\chi]\,x$.

(T3) Suppose $\eta([\mathsf{S}'{:}\,\chi']\,\varphi) \in [\mathsf{S}{:}\,\chi]\,x$ for every $\chi' \in \mathcal{L}$; then, $[\mathsf{S}{:}\,\chi]\,\eta([\mathsf{S}'{:}\,\chi']\,\psi) \in x$ for every $\chi' \in \mathcal{L}$. Being a theory, x is closed under $\mathsf{R}_{\mathsf{S}:*}$; moreover, $[\mathsf{S}{:}\,\chi]\,\eta(\sharp)$ is a necessity form. Hence, $[\mathsf{S}{:}\,\chi]\,\eta([\mathsf{S}'{:}\,*]\,\varphi) \in x$ and thus, by definition, $\eta([\mathsf{S}'{:}\,*]\,\varphi) \in [\mathsf{S}{:}\,\chi]\,x$. ■

Here are two further useful properties of theories that can be proven similarly to, e.g., Lemma 8 and Proposition 15 of Galimullin 2021.

Lemma 2 *If x is a theory, then $\varphi \in \varphi \to x$ and $x \subseteq \varphi \to x$.* ■

Lemma 3 *Let φ be a formula and x be a theory. Then $\varphi \to x$ is consistent if and only if $\neg\varphi \notin x$.* ■

Theories share some properties with maximal consistent sets.

Lemma 4 *Every consistent theory can be extended to a maximal consistent one.*

Proof. Let x be a consistent theory; let $\{\psi_0, \psi_1, \ldots\}$ be an enumeration of the $\mathcal{L}^*_{\mathsf{PC}}$-formulas. The maximal consistent theory y is built inductively. First, take $y_0 := x$. Then, given a consistent theory y_n satisfying $x \subseteq y_n$, consider the nth formula of the enumeration, ψ_n.

- If $\neg\psi_n \notin y_n$, then define $y_{n+1} := \psi_n \to y_n$.
- If $\neg\psi_n \in y_n$, consider two cases.
  - if $\neg\psi_n$ is *not* of the form $\neg\eta([\mathsf{S}{:}\,*]\,\varphi)$, then define $y_{n+1} := y_n$.
  - if $\neg\psi_n$ is of the form $\neg\eta([\mathsf{S}{:}\,*]\,\varphi)$, then define $y_{n+1} := \neg\eta([\mathsf{S}{:}\,\chi]\,\varphi) \to y_n$, with $\neg\eta([\mathsf{S}{:}\,\chi]\,\varphi)$ being the first formula in the enumeration that is not in y_n.

From its definition, y_{n+1} is a theory such that $y_n \subseteq y_{n+1}$ (Lemma 2). From its construction and Lemma 3, it is consistent.

Now, take $y := \bigcup_{n\in\mathbb{N}} y_n$. Its consistency follows from the consistency of all y_n. Moreover: it is a theory as it satisfies **(T1)** , **(T2)** and **(T3)** . The first follows because $x \subseteq y$ and x is a theory. The second is straightforward. For the third, consider its two cases.

- If $\neg\eta([\mathsf{S}{:}\,*]\,\varphi) \notin y_n$, then $\eta([\mathsf{S}{:}\,*]\,\varphi) \in y_{n+1}$ and therefore $\eta([\mathsf{S}{:}\,*]\,\varphi) \in y$. But $\mathcal{APC} \subset y$ so, by axiom $\mathsf{A}_{\mathsf{S}:*}$ and closure under MP, it follows that $\eta([\mathsf{S}{:}\,\chi]\,\varphi) \in y$ for all $\chi \in \mathcal{L}$.
- If $\neg\eta([\mathsf{S}{:}\,*]\,\varphi) \in y_n$ then, by construction, there is a χ such that $\neg\eta([\mathsf{S}{:}\,\chi]\,\varphi) \in y_{n+1}$. so $\neg\eta([\mathsf{S}{:}\,\chi]\,\varphi) \in y$. Then, by y's consistency, $\eta([\mathsf{S}{:}\,\chi]\,\varphi) \notin y$.

[19] In fact, the equivalence is derivable. By propositional reasoning, $\vdash (\varphi \to \psi) \leftrightarrow \neg(\varphi \wedge \neg\psi)$, so rule $\mathsf{RE}_{\mathsf{S}:\chi}$ yields $\vdash [\mathsf{S}{:}\,\chi](\varphi \to \psi) \leftrightarrow [\mathsf{S}{:}\,\chi]\,\neg(\varphi \wedge \neg\psi)$. From axiom $\mathsf{A}^{\neg}_{\mathsf{S}:\chi}$ one gets $\vdash [\mathsf{S}{:}\,\chi]\,\neg(\varphi \wedge \neg\psi) \leftrightarrow \neg\,[\mathsf{S}{:}\,\chi](\varphi \wedge \neg\psi)$. From axiom $\mathsf{A}^{\wedge}_{\mathsf{S}:\chi}$ one gets $\vdash [\mathsf{S}{:}\,\chi](\varphi \wedge \neg\psi) \leftrightarrow ([\mathsf{S}{:}\,\chi]\,\varphi \wedge [\mathsf{S}{:}\,\chi]\,\neg\psi)$ and thus, by propositional reasoning, $\vdash \neg\,[\mathsf{S}{:}\,\chi](\varphi \wedge \neg\psi) \leftrightarrow \neg([\mathsf{S}{:}\,\chi]\,\varphi \wedge [\mathsf{S}{:}\,\chi]\,\neg\psi)$. Axiom $\mathsf{A}^{\neg}_{\mathsf{S}:\chi}$ also produces $\vdash [\mathsf{S}{:}\,\chi]\,\neg\psi \leftrightarrow \neg\,[\mathsf{S}{:}\,\chi]\,\psi$, which via propositional reasoning can be turned into $\vdash ([\mathsf{S}{:}\,\chi]\,\varphi \wedge [\mathsf{S}{:}\,\chi]\,\neg\psi) \leftrightarrow ([\mathsf{S}{:}\,\chi]\,\varphi \wedge \neg\,[\mathsf{S}{:}\,\chi]\,\psi)$ and then into $\vdash \neg([\mathsf{S}{:}\,\chi]\,\varphi \wedge [\mathsf{S}{:}\,\chi]\,\neg\psi) \leftrightarrow \neg([\mathsf{S}{:}\,\chi]\,\varphi \wedge \neg\,[\mathsf{S}{:}\,\chi]\,\psi)$. Finally, propositional reasoning also produces $\vdash \neg([\mathsf{S}{:}\,\chi]\,\varphi \wedge \neg\,[\mathsf{S}{:}\,\chi]\,\psi) \leftrightarrow ([\mathsf{S}{:}\,\chi]\,\varphi \to [\mathsf{S}{:}\,\chi]\,\psi)$. From the five pieces and propositional reasoning, one gets $\vdash [\mathsf{S}{:}\,\chi](\varphi \to \psi) \leftrightarrow ([\mathsf{S}{:}\,\chi]\,\varphi \to [\mathsf{S}{:}\,\chi]\,\psi)$.

It is only left to show that y is maximal. Take any ψ_n in the enumeration. If $\neg\psi_n \notin y_n$ then, by construction, $\psi_n \in y_{n+1}$ (Lemma 2) and thus $\psi_n \in y$. Otherwise, $\neg\psi_n \in y_n$ so $\neg\psi_n \in y$. ■

One can now define the canonical pseudo-model.

Definition A.4 The canonical pseudo-model $\mathbb{M}$ is the tuple $\langle \mathbb{W}, \mathbb{R}, \mathbb{V} \rangle$ where

$$\begin{aligned} \mathbb{W} &:= \{x \mid x \text{ is a maximal consistent theory}\}, \\ \mathbb{R}_{\mathtt{G}} &:= \{(x, y) \subseteq \mathbb{W} \times \mathbb{W} \mid \mathrm{D}_{\mathtt{H}}\, x \subseteq y \text{ for all } \varnothing \subset \mathtt{H} \subseteq \mathtt{G}\}, \\ \mathbb{V}(p) &:= \{x \in \mathbb{W} \mid p \in x\}. \end{aligned}$$ ◀

The following lemma plays the role of the existence lemma.

Lemma 5 *Let x be a theory. If $\mathrm{D}_{\mathtt{G}}\, \varphi \notin x$, then there is a maximal consistent theory y such that $\mathrm{D}_{\mathtt{G}}\, x \subseteq y$ and $\varphi \notin y$.*

Proof. Suppose $\mathrm{D}_{\mathtt{G}}\, \varphi \notin x$. By definition, $\varphi \notin \mathrm{D}_{\mathtt{G}}\, x$, so $\neg\varphi \to \mathrm{D}_{\mathtt{G}}\, x$ is a consistent theory (Lemma 3); by Lemma 4, it can be extended into a maximal consistent theory y containing $\neg\varphi$. Then, from its consistency, y does not contain φ. ■

Finally, the truth lemma.

Lemma 6 *For every $\phi \in \mathcal{L}^{*}_{\mathsf{PC}}$ and every maximal consistent theory x,*

$$(\mathbb{M}, x) \Vdash \phi \quad \textit{if and only if} \quad \phi \in x$$

Proof. The proof proceeds by structural induction on ϕ.

- **Base case** p. By the definition of the valuation, as $x \in \mathbb{V}(p)$ iff $p \in x$.

For the inductive cases, the IH states that $(\mathbb{M}, x) \Vdash \psi$ iff $\psi \in x$ holds for all maximal consistent theories x and formulas ψ such that $\psi \prec \phi$.

- **Inductive cases** $\neg\varphi$**,** $\varphi \wedge \psi$. Straightforward.

- **Inductive case** $\mathrm{D}_{\mathtt{G}}\, \varphi$. **(⇒)** For a contraposition argument, suppose $\mathrm{D}_{\mathtt{G}}\, \varphi \notin x$. Then, by Lemma 5, there is a $y \in \mathbb{W}$ such that both $\mathrm{D}_{\mathtt{G}}\, x \subseteq y$ and $\varphi \notin y$. By the definition of $\mathbb{R}_{\mathtt{G}}$ and IH, this means that there is a $y \in \mathbb{W}$ such that both $\mathbb{R}_{\mathtt{G}}xy$ and $(\mathbb{M}, y) \nVdash \varphi$. Therefore, by semantic interpretation, $(\mathbb{M}, x) \nVdash \mathrm{D}_{\mathtt{G}}\, \varphi$. **(⇐)** Suppose $\mathrm{D}_{\mathtt{G}}\, \varphi \in x$; take any $y \in \mathbb{W}$ such that $\mathbb{R}_{\mathtt{G}}xy$. From $\mathrm{D}_{\mathtt{G}}\, \varphi \in x$ it follows that $\varphi \in \mathrm{D}_{\mathtt{G}}\, x$; from $\mathbb{R}_{\mathtt{G}}xy$ it follows that $\mathrm{D}_{\mathtt{G}}\, x \subseteq y$. From these two pieces, $\varphi \in y$; thus, by IH, $(\mathbb{M}, y) \Vdash \varphi$. So, every $y \in \mathbb{W}$ with $\mathbb{R}_{\mathtt{G}}xy$ is such that $(\mathbb{M}, y) \Vdash \varphi$; hence, $(\mathbb{M}, x) \Vdash \mathrm{D}_{\mathtt{G}}\, \varphi$.

- **Inductive cases** $[\mathsf{S}{:}\,\chi]\, p$**,** $[\mathsf{S}{:}\,\chi]\, \neg\varphi$**,** $[\mathsf{S}{:}\,\chi](\varphi \wedge \psi)$**,** $[\mathsf{S}{:}\,\chi]\, \mathrm{D}_{\mathtt{G}}\, \varphi$ **and** $[\mathsf{S}{:}\,\chi]\, [\mathsf{S}'{:}\,\chi']\, \varphi$. They are all handled using the axioms and rule in Table 2 (see Velázquez-Quesada 2022 for a similar proof detailing how they are used). Here, just the case for $[\mathsf{S}{:}\,\chi]\, \mathrm{D}_{\mathtt{G}}\, \varphi$ is (briefly) discussed. From the soundness of axiom $\mathsf{A}^{\mathrm{D}}_{\mathsf{S}:\chi}$, it follows that $(\mathbb{M}, x) \Vdash [\mathsf{S}{:}\,\chi]\, \mathrm{D}_{\mathtt{G}}\, \varphi$ if and only if $(\mathbb{M}, x) \Vdash \mathrm{D}_{\mathsf{S} \cup \mathtt{G}}\, [\mathsf{S}{:}\,\chi]\, \varphi \wedge \mathrm{D}^{\chi}_{\mathtt{G}}\, [\mathsf{S}{:}\,\chi]\, \varphi$. But the complexity among the formulas inside the scope of $[\mathsf{S}{:}\,\chi]$ has decreased, so $\mathrm{D}_{\mathsf{S} \cup \mathtt{G}}\, [\mathsf{S}{:}\,\chi]\, \varphi \wedge \mathrm{D}^{\chi}_{\mathtt{G}}\, [\mathsf{S}{:}\,\chi]\, \varphi \prec [\mathsf{S}{:}\,\chi]\, \mathrm{D}_{\mathtt{G}}\, \varphi$; thus, by IH, $\mathrm{D}_{\mathsf{S} \cup \mathtt{G}}\, [\mathsf{S}{:}\,\chi]\, \varphi \wedge \mathrm{D}^{\chi}_{\mathtt{G}}\, [\mathsf{S}{:}\,\chi]\, \varphi \in x$. Now, x is a theory, so it contains $\mathcal{APC}$ and thus, in particular, it contains (all instances of) axiom $\mathsf{A}^{\mathrm{D}}_{\mathsf{S}:\chi}$ and is closed under MP. Hence, $\mathrm{D}_{\mathsf{S} \cup \mathtt{G}}\, [\mathsf{S}{:}\,\chi]\, \varphi \wedge \mathrm{D}^{\chi}_{\mathtt{G}}\, [\mathsf{S}{:}\,\chi]\, \varphi \in x$ if and only if $[\mathsf{S}{:}\,\chi]\, \mathrm{D}_{\mathtt{G}}\, \varphi \in x$.

- **Inductive case** $[\mathsf{S}\colon\chi]\,[\mathsf{S}'\colon *]\,\varphi$. **(⇒)** Suppose $(\mathbb{M}, x) \Vdash [\mathsf{S}\colon\chi]\,[\mathsf{S}'\colon *]\,\varphi$, so $(\mathbb{M}, x) \Vdash [\mathsf{S}\colon\chi]\,[\mathsf{S}'\colon\chi']\,\varphi$ for all $\chi' \in \mathcal{L}$. But $[\mathsf{S}\colon\chi]\,[\mathsf{S}'\colon\chi']\,\varphi \prec [\mathsf{S}\colon\chi]\,[\mathsf{S}'\colon *]\,\varphi$; hence, from IH, it follows that $[\mathsf{S}\colon\chi]\,[\mathsf{S}'\colon\chi']\,\varphi \in x$, for all $\chi' \in \mathcal{L}$. Now, $[\mathsf{S}\colon\chi]\,\eta(\sharp)$ is a necessity form; since x is closed under rule $\mathsf{R}_{\mathsf{S}\colon *}$, it follows that $[\mathsf{S}\colon\chi]\,[\mathsf{S}'\colon *]\,\varphi \in x$. **(⇐)** Suppose $[\mathsf{S}\colon\chi]\,[\mathsf{S}'\colon *]\,\varphi \in x$. Since x is a theory, $[\mathsf{S}\colon\chi]\,[\mathsf{S}'\colon\chi']\,\varphi \in x$, for all $\chi' \in \mathcal{L}$. Once again, $[\mathsf{S}\colon\chi]\,[\mathsf{S}'\colon\chi']\,\varphi \prec [\mathsf{S}\colon\chi]\,[\mathsf{S}'\colon *]\,\varphi$, so from IH it follows that $(\mathbb{M}, x) \Vdash [\mathsf{S}\colon\chi]\,[\mathsf{S}'\colon\chi']\,\varphi$ holds for all $\chi' \in \mathcal{L}$, that is, $(\mathbb{M}_{\mathsf{S}\colon\chi}, x) \Vdash [\mathsf{S}'\colon\chi']\,\varphi$ holds for all $\chi' \in \mathcal{L}$. By semantic interpretation, this is equivalent to $(\mathbb{M}_{\mathsf{S}\colon\chi}, x) \Vdash [\mathsf{S}'\colon *]\,\varphi$, and thus to $(\mathbb{M}, x) \Vdash [\mathsf{S}\colon\chi]\,[\mathsf{S}'\colon *]\,\varphi$.
- **Inductive case** $[\mathsf{S}\colon *]\,\varphi$. **(⇒)** Suppose $(\mathbb{M}, x) \Vdash [\mathsf{S}\colon *]\,\varphi$. Then, $(\mathbb{M}, x) \Vdash [\mathsf{S}\colon\chi]\,\varphi$ for all $\chi \in \mathcal{L}$. But $[\mathsf{S}\colon\chi]\,\varphi \prec [\mathsf{S}\colon *]\,\varphi$ so, from IH, $[\mathsf{S}\colon\chi]\,\varphi \in x$ for all $\chi \in \mathcal{L}$. Since x is closed under rule $\mathsf{R}_{\mathsf{S}\colon *}$, it follows that $[\mathsf{S}'\colon *]\,\varphi \in x$. **(⇐)** Suppose $[\mathsf{S}'\colon *]\,\varphi \in x$. From axiom $\mathsf{A}_{\mathsf{S}\colon *}$ and x's closure under MP we have $[\mathsf{S}\colon\chi]\,\varphi \in x$ for all $\chi \in \mathcal{L}$. But $[\mathsf{S}\colon\chi]\,\varphi \prec [\mathsf{S}\colon *]\,\varphi$ so, from IH, $(\mathbb{M}, x) \Vdash [\mathsf{S}\colon\chi]\,\varphi$ for all $\chi \in \mathcal{L}$. Thus, $(\mathbb{M}, x) \Vdash [\mathsf{S}\colon *]\,\varphi$. ■

With the canonical pseudo-model $\mathbb{M}$ satisfying this truth lemma, the final stage of the proof consists in creating a collectively bisimilar structure (so it agrees with $\mathbb{M}$ in the satisfiability of formulas in $\mathcal{L}^*_{\mathsf{PC}}$) that is, additionally, a *model*. The new structure is a tree-like model $\mathtt{M}$ obtained by unravelling $\mathbb{M}$ around every world in its domain $\mathbb{W}$. As a result, $\mathtt{M}$ has a forest structure with no unique root.

Definition A.5 The tree-like canonical model $\mathtt{M}$ is the tuple $\langle \mathtt{W}, \mathtt{R}, \mathtt{V} \rangle$ where

- $\mathtt{W}$ is the set of all finite paths $\mathtt{x} = \langle x_0 \cdot \mathtt{G}_1 \cdot x_1 \cdot \ldots \cdot \mathtt{G}_n \cdot x_n \rangle$ such that x_k is in $\mathbb{W}$ for every $[0\,..\,n]$ and $\mathbb{R}_{\mathtt{G}_{k+1}} x_k x_{k+1}$ for every $k \in [0\,..\,n-1]$. The last world in a path $\mathtt{x}$ is denoted as $\mathrm{last}(\mathtt{x})$.
- $\mathtt{R} = \{\mathtt{R}_\mathtt{i} \subseteq \mathtt{W} \times \mathtt{W} \mid \mathtt{i} \in \mathtt{A}\}$ with $\mathtt{R}_\mathtt{i} := \{(\mathtt{x}, \mathtt{y}) \mid \mathtt{y} = \langle \mathtt{x} \cdot \mathtt{G} \cdot \mathrm{last}(\mathtt{y}) \rangle \text{ and } \mathtt{i} \in \mathtt{G}\}$. Write $\mathtt{R}_\mathtt{G}$ for $\bigcap_{\mathtt{i} \in \mathtt{G}} \mathtt{R}_\mathtt{i}$.
- For all $p \in \mathtt{P}$, $\mathtt{V}(p) := \{\mathtt{x} \in \mathtt{W} \mid p \in \mathrm{last}(\mathtt{x})\}$. ◀

Note how $\mathtt{M}$ is a model. More importantly: concerning the satisfiability of formulas in $\mathcal{L}^*_{\mathsf{PC}}$, it is just as $\mathbb{M}$. Since every model is a pseudo-model, the following lemma will treat $\mathtt{M}$ and $\mathbb{M}$ as pseudo-models.

Lemma 7 *The structures* $\mathtt{M} = \langle \mathtt{W}, \mathtt{R}, \mathtt{V} \rangle$ *and* $\mathbb{M} = \langle \mathbb{W}, \mathbb{R}, \mathbb{V} \rangle$ *are collectively bisimilar.*

Proof. Define the following relation, connecting each theory $x \in \mathbb{W}$ with every path in $\mathtt{W}$ whose last world is x:

$$Z = \{(x, \mathtt{x}) \mid x = \mathrm{last}(\mathtt{x})\}.$$

To show that Z is a collective bisimulation, take any $(x, \mathtt{x}) \in Z$.

- **Atoms**. For every atom p we have $x \in \mathbb{V}(p)$ iff $p \in x$ (definition of $\mathbb{V}$) iff $p \in \mathrm{last}(\mathtt{x})$ (as $x = \mathrm{last}(\mathtt{x})$, by definition of Z) iff $\mathtt{x} \in \mathtt{V}(p)$ (definition of $\mathtt{V}$).
- **Forth**. Take any $\mathtt{G} \subseteq \mathtt{A}$ and any $y \in \mathbb{W}$ such that $\mathbb{R}_\mathtt{G} xy$. Because of $x = \mathrm{last}(\mathtt{x})$ and $\mathbb{R}_\mathtt{G} xy$, the path $\mathtt{x}$ can be extended into the path $\mathtt{y} = \langle \mathtt{x} \cdot \mathtt{G} \cdot y \rangle$; since y is the last world in this path, we actually have $\mathtt{y} = \langle \mathtt{x} \cdot \mathtt{G} \cdot \mathrm{last}(\mathtt{y}) \rangle$. From the definition of $\mathtt{R}$, it follows that $\mathtt{R}_\mathtt{G} \mathtt{x}\mathtt{y}$. Finally, we have $y = \mathrm{last}(\mathtt{y})$, so $(y, \mathtt{y}) \in Z$.

- **Back**. Take any $\mathtt{G} \subseteq \mathtt{A}$ and any $\mathtt{y} \in \mathtt{W}$ such that $\mathtt{R_G xy}$. From the definition of $\mathtt{R}$, it follows that $\mathtt{y} = \langle \mathtt{x}{\cdot}\mathtt{G}{\cdot}y \rangle$ for some world y. Then, by the definition of a path, $\mathbb{R}_{\mathtt{G}}\, \mathrm{last}(\mathtt{x})y$. Finally, it is clear that $y = \mathrm{last}(\mathtt{y})$, so $(y, \mathtt{y}) \in Z$.

It is only left to show that Z is non-empty, for which it is enough to notice that every theory $x \in \mathbb{W}$ has a matching path $\mathtt{x} = \langle x \rangle$, which clearly satisfies $\mathrm{last}(\mathtt{x}) = x$. ■

Finally, for completeness, one argues that every valid formula is derivable in the system, which is equivalent to saying that every valid formula is in $\mathcal{APC}$.

Theorem 14 *Every valid formula in $\mathcal{L}^*_{\mathsf{PC}}$ is in $\mathcal{APC}$.*

Proof. For a contradiction, suppose there is a valid φ such that $\varphi \notin \mathcal{APC}$. Build the theory $\neg\varphi \to \mathcal{APC}$ which, by Lemma 3, is consistent and, by Lemma 2, contains $\neg\varphi$. Then, by Lemma 4, $\neg\varphi \to \mathcal{APC}$ can be extended into a maximal consistent theory x such that $\neg\varphi \to \mathcal{APC} \subseteq x$. Moreover: since x is consistent and it contains $\neg\varphi$, we have $\varphi \notin x$. But then, $(\mathbb{M}, x) \nVdash \varphi$ (by Lemma 6) and thus $(\mathtt{M}, \langle x \rangle) \nVdash \varphi$ (from Lemma 7 and Theorem 9). Hence, φ is false in some model, contradicting the fact that it is valid. ■

Proof of Theorem 9

Since $\mathcal{L}^*_{\mathsf{PC}}$ is the union of $\mathcal{L}^*_{\mathsf{PC}}[n]$ for all $n \in \mathbb{N}$, proceed again by induction on n (as in the proof of Theorem 4). Again, one proves a stronger statement: for every $\psi \in \mathcal{L}^*_{\mathsf{PC}}$ with $\mathrm{at}(\psi) \subseteq \mathtt{Q}$ and every (M, w) and (M', w'), if $(M, w) \rightleftarrows^{\mathtt{Q}}_{C} (M', w')$ then ***(1)*** $(M, w) \Vdash \psi$ if and only if $(M', w') \Vdash \psi$, and **(2)** $(M_{\mathtt{S}:\psi}, w) \rightleftarrows^{\mathtt{Q}}_{C} (M'_{\mathtt{S}:\psi}, w')$. Thus, take $M = \langle W, R, V \rangle$ and $M' = \langle W', R', V' \rangle$.

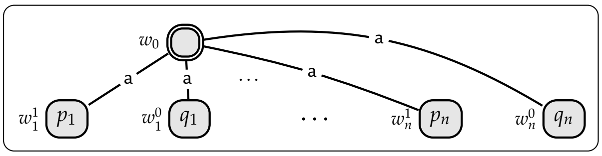
Base case. This base case is for formulas in $\mathcal{L}^*_{\mathsf{PC}}[0]$, defined as the basic language $\mathcal{L}$ plus the modality $[\mathtt{S}{:}\,*]$. For Item ***(1)***, proceed by structural induction. The cases for formulas in $\mathcal{L}$ (atoms, Boolean operators and $\mathrm{D_G}$) are covered by Theorem 2. For the remaining case, take $[\mathtt{S}{:}\,*]\,\varphi$ with $\varphi \in \mathcal{L}$ and $\mathrm{at}([\mathtt{S}{:}\,*]\,\varphi) = \mathrm{at}(\varphi) \subseteq \mathtt{Q}$; suppose $(M, w) \rightleftarrows^{\mathtt{Q}}_{C} (M', w')$. From left to right, if $(M, w) \Vdash [\mathtt{S}{:}\,*]\,\varphi$ then, by semantic interpretation, $(M_{\mathtt{S}:\chi}, w) \Vdash \varphi$ holds for every $\chi \in \mathcal{L}$. But from $(M, w) \rightleftarrows^{\mathtt{Q}}_{C} (M', w')$ and the fact each χ is in $\mathcal{L}$, it follows that $(M_{\mathtt{S}:\chi}, w) \rightleftarrows^{\mathtt{Q}}_{C} (M'_{\mathtt{S}:\chi}, w')$ for every $\chi \in \mathcal{L}$ (essentially Item ***(2)*** in the base case of the proof of Theorem 4, as the proof also works for any χ, regardless of the atoms it contains). Then, from IH and $\mathrm{at}(\varphi) \subseteq \mathtt{Q}$, it follows that $(M'_{\mathtt{S}:\chi}, w') \Vdash \varphi$ for every $\chi \in \mathcal{L}$; hence, $(M', w') \Vdash [\mathtt{S}{:}\,*]\,\varphi$. The right-to-left direction is analogous. For Item ***(2)***, proceed as in the same case in the proof of Theorem 4, using now the just proved Item ***(1)*** for formulas in $\mathcal{L}^*_{\mathsf{PC}}[0]$.

Inductive case. As in the same case in the proof of Theorem 4.

Proof of Theorem 11

Constructing restrictions takes polynomial time (due to polynomial time construction of the bisimulation contraction; see, e.g., Kanellakis and Smolka 1990) and thus polynomial space. The space required for the $[\mathtt{S}{:}\,\chi]\,\psi$ case is bounded by $O(|\varphi| \cdot |M|)$. For the $[\mathtt{S}{:}\,*]\,\psi$ case, collective bisimulation contraction can be computed in polynomial time and space, and each restriction has a size of at most $|M|$. If one traverses a given formula depth-first and reuses memory, the space to store model restrictions is polynomial in $|\varphi|$ (even though the algorithm itself runs in exponential time). Thus, the space required for the case of $[\mathsf{S}:*]\,\psi$ is bounded by $O(|\varphi| \cdot |M|)$. Finally, since computing each subformula of φ requires space bounded by $O(|\varphi| \cdot |M|)$, the space required by the whole algorithm is bounded by $O(|\varphi|^2 \cdot |M|)$. The algorithm follows closely the semantics of $\mathcal{L}^*_{\mathsf{PC}}$, and correctness can be shown via induction on φ. For the case of quantifiers note that, to switch from bipartitions to particular formulas corresponding to those partitions, one can use characteristic formulas (van Ditmarsch et al. 2014), which are built in such a way that they are true only in one world of a model (up to collective bisimilarity).

To show that the model checking problem is *PSPACE*-hard, use the classic reduction from the satisfiability of QBF. Without loss of generality, consider QBFs without free variables in which every variable is quantified only once. Consider a QBF with n variables $\{x_1, \ldots, x_n\}$. We need a formula in $\mathcal{L}^*_{\mathsf{PC}}$ and a model with the size of both being polynomial on the size of the QBF. The (reflexive and symmetric) model M^n below satisfies this: w_0 is the evaluation point, and for each variable x_i there are two worlds, w_i^1 and w_i^0, corresponding respectively to evaluating x_i to 1 and to 0. Assume that each w_i^1 satisfies only p_i and each w_i^0 satisfies only q_i. Observe that R_b is just the identity.



Let $\Psi := Q_1 x_1 \ldots Q_n x_n \Phi(x_1, \ldots, x_n)$ be a quantified Boolean formula (so $Q_i \in \{\forall, \exists\}$ and $\Phi(x_1, \ldots, x_n)$ is Boolean). The formula $chosen_k$ below indicates, intuitively, that the values (either 1 or 0) of the first k variables have been chosen (thus, for $1 \leqslant i \leqslant k$, exactly one world in $\{w_i^1, w_i^0\}$ can be accessed from w).

$$chosen_k := \bigwedge_{1 \leqslant i \leqslant k} (\widehat{\mathrm{K}}_\mathsf{a}\, p_i \leftrightarrow \neg\, \widehat{\mathrm{K}}_\mathsf{a}\, q_i) \wedge \bigwedge_{k < i \leqslant n} (\widehat{\mathrm{K}}_\mathsf{a}\, p_i \wedge \widehat{\mathrm{K}}_\mathsf{a}\, q_i).$$

Here is, then, a recursive translation from Ψ to a formula ψ in $\mathcal{L}^*_{\mathsf{PC}}$:

$$\begin{aligned} \psi_0 &:= \Phi(\widehat{\mathrm{K}}_\mathsf{a}\, p_1, \ldots, \widehat{\mathrm{K}}_\mathsf{a}\, p_n), \\ \psi_k &:= \begin{cases} [\{\mathsf{a},\mathsf{b}\}:*](chosen_k \rightarrow \psi_{k-1}) & \text{if } Q_k = \forall \\ \langle\{\mathsf{a},\mathsf{b}\}:*\rangle(chosen_k \wedge \psi_{k-1}) & \text{if } Q_k = \exists \end{cases}, \\ \psi &:= \psi_n. \end{aligned}$$

Now, we need to show that

$$Q_1 x_1 \ldots Q_n x_n \Phi(x_1, \ldots, x_n) \text{ is satisfiable} \quad \text{if and only if} \quad (M^n, w_0) \Vdash \psi.$$

For this, observe that each world in M^n can be characterised by a unique formula. Moreover, relation b is the identity. Therefore, $[\{\mathsf{a},\mathsf{b}\}:*]$ and $\langle\{\mathsf{a},\mathsf{b}\}:*\rangle$ can force any restriction of the a-edges from w_0 to w_i's. In the model, worlds

w_i^1 and w_i^0 correspond to the truth-value of x_i. The guard $chosen_k$ guarantees that only the truth-values of the first k variables have been chosen, and that they have been chosen unambiguously (i.e. there is exactly one edge from w_0 to either w_i^1 and w_i^0). Thus, together with $[\{\mathsf{a},\mathsf{b}\}:*]$ and $\langle\{\mathsf{a},\mathsf{b}\}:*\rangle$, the guards $chosen_k$ emulate $\forall$ and $\exists$. Then, once the values of all x_i's have been set, the evaluation of the QBF corresponds to the a-reachability of the corresponding worlds in M^n.

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