Quantifying Over Information Change with Common Knowledge*

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Abstract

Public announcement logic (PAL) extends multi-agent epistemic logic with dynamic operators modelling the effects of public communication. Allowing quantification over public announcements lets us reason about the *existence* of an announcement that reaches a certain epistemic goal. Two notable examples of logics of quantified announcements are arbitrary public announcement logic (APAL) and group announcement logic (GAL). While the notion of common knowledge plays an important role in PAL, and in particular in characterisations of epistemic states that an agent or a group of agents might make come about by performing public announcements, extensions of APAL and GAL with common knowledge still haven't been studied in detail. That is what we do in this paper. In particular, we consider both conservative extensions, where the semantics of the quantifiers is not changed, as well as extensions where the scope of quantification also includes common knowledge formulas. We compare the expressivity of these extensions relative to each other and other connected logics, and provide sound and complete axiomatisations. Finally, we show how the completeness results can be used for other logics with quantification over information change.

1 Introduction

Quantified dynamic epistemic logics and common knowledge *Epistemic logic* (EL) [34] is a multimodal logic where formulas $\Box_a\varphi$ mean 'agent a knows φ'. Formulas of EL are interpreted on epistemic models that consist of states and equivalence relations between them for each agent. Such a logic allows us to reason not only about an agent's knowledge of some basic facts, but about what other agents know as well.

While EL deals with *individual* knowledge of particular agents, there are also various kinds of *group* knowledge. A prime example of group knowledge is *common knowledge*

*Extended version of [28]. The main extension is the other direction of the GALC$\setminus$GALC^X and APALC$\setminus$APALC^X expressivity result. Section 6 is completely new.

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which has played an important part in reasoning about knowledge in the multi-agent setting [24]. It has also been used in epistemic planning [37], machine learning [42], game theory [38], and so on. *Epistemic logic* (EL) *with common knowledge* (ELC) [24] extends the language of EL with common knowledge modalities $\blacksquare_G\varphi$, where G is a subset of the set of all agents. Informally, $\blacksquare_G\varphi$ is read as 'everybody in G knows that φ, everybody in G knows that everybody in G knows that φ, and so on'. On the level of models this corresponds to truth in all states accessible by the reflexive transitive closure of relations for agents from G.

Both EL and ELC provide a static description of knowledge in a multi-agent system. Logics that are covered by the umbrella term *dynamic epistemic logic* (DEL) [20] study the effects of various epistemic events on the individual and group knowledge of agents. The prime example of such a logic is *public announcement logic* (PAL) [41] that models public communication. A public announcement is an event where all agents publicly and simultaneously receive the same true piece of information. Syntactically, PAL extends EL with construct $[\psi]\varphi$ that say 'after truthful public announcement of ψ, φ is true'. From the model perspective, public announcement of ψ removes all the states from a model that do not satisfy ψ. The interaction of epistemic events, in particular of public announcements, and common knowledge was studied in [11].

Aribitrary public announcement logic (APAL) [9] and *group announcement logic* (GAL) [2] are extensions of PAL with quantifiers over possible truthful announcements. APAL extends PAL with constructs of the form $\langle!\rangle\varphi$ that mean 'after *some* public announcement, φ holds'. GAL has quantifiers with a more limited scope, with group announcement operators $\langle G\rangle\varphi$ meaning that '*after some* (joint) announcement by agents from group G, φ is true'. A 'joint announcement' in this context means an announcement of a formula of the shape $\bigwedge_{i\in G}\Box_i\varphi_i$. In other words, each agent can announce something they know. GAL thus allows us to reason about the *ability* of an agent or a group of agents to achieve their epistemic goal by a joint public announcement.

Common knowledge plays a significant role in PAL, and in particular in characterisations of epistemic states that an agent or a group of agents might make come about by making public announcements. Investigating logics of quantified announcements (or any other quantified epistemic actions) with common knowledge is long overdue, and it was reiterated as an open question in a recent survey [14]. In this paper, we address this problem. First, we study the languages APALC and GALC obtained by extending APAL and GAL, respectively, with common knowledge without changing the semantics of any of the operators. This allows us to gain further insight into the standard APAL and GAL modalities. There is a subtlety here, however, in the scope of quantification. In both APAL and GAL the quantification is restricted to announcements in the purely epistemic language. The reason for this is, in addition to the fact that the quantification does not range over formulas with quantifiers in them to avoid circularity, that EL and PAL are equally expressive [41]. Thus quantifying over EL has the same effect as quantifying over PAL. Adding common knowledge changes the picture, since EL and ELC are not equally expressive. In this paper, in addition to the 'conservative' variants APALC and GALC, we also study variants of APAL and GAL with common knowledge where the quantification ranges over formulas of ELC, called APALC^X and GALC^X (for 'eXtended semantics'), respectively. It turns out that the difference in scope of the quantifiers is significant and non-trivial.

Overview of the paper and main results In Section 2 we introduce languages of the logics and the corresponding semantics. We investigate some intuitive potential properties of the interaction between quantified announcements and common knowledge in Section 3. In particular, we show that some of the immediate intuitions about sharing knowledge in a group and between groups are actually not correct. Then we specify a fragment of the language for which these intuitions indeed hold.

Section 4 is devoted to the study of the relative expressivity of the languages of GALC, GALC^X, APALC, and APALC^X and situating these languages within a broader landscape of EL-based logics. We show that both pairs, APALC and APALC^X, and GALC and GALC^X, are in fact incomparable when it comes to the expressive power. The fact that GALC^X and APALC^X can express some properties of models that cannot be captured by GALC and APALC, respectively, perhaps follows intuition. The converse, however, may come across as unexpected. In the proof, we demonstrate that sometimes the existential quantification over announcements in GALC^X and APALC^X is 'too powerful' to notice a difference in models – even though the same announcement might not have the same effect in both models there is often another announcement in the scope of quantification that has the same effect.

In Section 5 we give sound and complete proof systems for APALC, GALC, APALC^X, and GALC^X. Like all existing complete systems for APAL and GAL, these are infinitary. A detailed proof is given for the case of GALC; the other cases follow by relatively simple modifications. Our treatment of common knowledge differs from the classic fixed-point approach. Since both APAL and GAL are already infinitary, we use a straightforward infinitary inference rule for common knowledge as well.

Our completeness proof is modular in its nature, meaning that the parts corresponding to common knowledge can be reused as is for other logics with *quantification over information change*. In Section 6 we show that the proof can be adapted to obtain axiomatisations and completeness results for two decidable restrictions of APAL[1] extended with common knowledge, namely Boolean APAL [16] and Positive APAL [19]. A similar result can also be obtained for a variant of *coalition announcement logic* (CAL) [3, 26] that is called *coalition and relativised group announcement logic* (CoRGAL) [27] extended with common knowledge. Coalition announcement modalities $[\langle G\rangle]\varphi$ quantify over announcements by agents from G and *simultaneous* counter announcement by the agents outside of G. These constructs are read as 'whatever agents from G announce, there is a simultaneous announcement by the agents from outside of G such that φ is true after the joint announcement'.

There are also logics that quantify over other types of information changing events (see [14]), for some of which only infinitary axiomatisations are known. We claim that

[1]Note that APAL itself, as well as GAL, is undecidable [4].

our completeness proof can be used to show the completeness of their extensions with common knowledge. As an example, we consider *arbitrary arrow update logic with common knowledge* and indicate how to obtain its complete axiomatisation.

Finally, we conclude in Section 7 and discuss directions of further research.

2 Logics of Quantified Announcements with Common Knowldge

2.1 Syntax and Semantics

Let us fix a finite set of agents A and a countable set of propositional variables P.

Definition 2.1. The *language of arbitrary public announcement logic with common knowledge* $\mathcal{APALC}$, the *language of group announcement logic with common knowledge* $\mathcal{GALC}$ and their *extended* versions $\mathcal{APALC}^X$ and $\mathcal{GALC}^X$ respectively, are inductively defined as

$$\begin{array}{ll} \mathcal{APALC} & \ni \varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box_a\varphi \mid [\varphi]\varphi \mid \blacksquare_G\varphi \mid [!]\varphi \\ \mathcal{GALC} & \ni \varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box_a\varphi \mid [\varphi]\varphi \mid \blacksquare_G\varphi \mid [G]\varphi \\ \mathcal{APALC}^X & \ni \varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box_a\varphi \mid [\varphi]\varphi \mid \blacksquare_G\varphi \mid [!]^X\varphi \\ \mathcal{GALC}^X & \ni \varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box_a\varphi \mid [\varphi]\varphi \mid \blacksquare_G\varphi \mid [G]^X\varphi \end{array}$$

where $p \in P$, $a \in A$, and $G \subseteq A$. Duals are defined as $\Diamond_a\varphi := \neg\Box_a\neg\varphi$, $\langle\psi\rangle\varphi := \neg[\psi]\neg\varphi$, $\blacklozenge_G\varphi := \neg\blacksquare_G\neg\varphi$, $\langle!\rangle\varphi := \neg[!]\neg\varphi$, $\langle!\rangle^X\varphi := \neg[!]^X\neg\varphi$, $\langle G\rangle\varphi := \neg[G]\neg\varphi$ and $\langle G\rangle^X\varphi := \neg[G]^X\neg\varphi$.

Formula $\Box_a\varphi$ is read as 'agent a knows φ'; $[\psi]\varphi$ means that 'after truthful public announcement of ψ, φ will hold'; $\blacksquare_G\varphi$ is read as 'it is common knowledge among agents from group G that φ'; $[!]\varphi$ and $[!]^X\varphi$ are read as 'after any truthful public announcement, φ holds'; $[G]\varphi$ and $[G]^X\varphi$ are read as 'after any truthful public announcement by agents from group G, φ holds';.

The fragment of $\mathcal{GALC}$ without $[G]\varphi$ is called *public announcement logic with common knowledge* $\mathcal{PALC}$; the latter without $[\varphi]\varphi$ is *epistemic logic with common knowledge* $\mathcal{ELC}$; $\mathcal{PALC}$ and $\mathcal{ELC}$ minus $\blacksquare_G\varphi$ are, correspondingly, *public announcement logic* $\mathcal{PAL}$ and *epistemic logic* $\mathcal{EL}$. Finally, fragments of $\mathcal{GALC}$ and $\mathcal{APALC}$ without $\blacksquare_G\varphi$ are called *group announcement logic* $\mathcal{GAL}$ and *arbitrary public announcement logic* $\mathcal{APAL}$ respectively.

'Everyone in group G knows φ' is denoted by $\Box_G\varphi := \bigwedge_{i\in G}\Box_i\varphi$, and $\Box^n_G\varphi$ is defined inductively as $\Box^0_G\varphi := \varphi$ and $\Box^{n+1}_G\varphi := \Box_G\Box^n_G\varphi$ for all natural numbers n. Expression $\Diamond^n_G\varphi$ is defined similarly by substituting diamonds instead of boxes.

Definition 2.2. *Modal depth* of $\varphi \in \mathcal{APALC} \cup \mathcal{GALC} \cup \mathcal{APALC}^X \cup \mathcal{GALC}^X$ (denoted $md(\varphi)$) is defined inductively as

$$md(p) = 0 \qquad\qquad md([\psi]\varphi) = md(\psi) + md(\varphi)$$

$$md(\neg\varphi) = md(\varphi) \qquad md(\varphi \wedge \psi) = \max(md(\varphi), md(\psi))$$
$$md(\Box_a\varphi) = md(\blacksquare_G\varphi) = md([!]\varphi) = md([G]\varphi) = md([!]^X\varphi) = md([G]^X\varphi) = md(\varphi) + 1$$

Definition 2.3. Let $\varphi \in \mathcal{APALC} \cup \mathcal{GALC} \cup \mathcal{APALC}^X \cup \mathcal{GALC}^X$. The *quantifier depth* $\delta_\forall(\varphi)$ of φ is defined inductively as

$$\delta_\forall(p) = 0 \qquad \delta_\forall([\psi]\varphi) = \delta_\forall(\psi) + \delta_\forall(\varphi)$$
$$\delta_\forall(\neg\varphi) = \delta_\forall(\Box_a\varphi) = \delta_\forall(\varphi) \qquad \delta_\forall(\blacksquare_G\varphi) = \delta_\forall(\varphi)$$
$$\delta_\forall(\varphi \wedge \psi) = \max(\delta_\forall(\varphi), \delta_\forall(\psi)) \qquad \delta_\forall(Q\varphi) = \delta_\forall(\varphi) + 1$$

where $Q \in \{[!], [G], [!]^X, [G]^X\}$.

Definition 2.4. A *model* M is a tuple (S, R, V), where S is a non-empty set of states, $R : A \to 2^{S\times S}$ gives an equivalence relation for each agent, and $V : P \to 2^S$ is the valuation function. We will denote model M with a distinguished state s as M_s. Whenever necessary, we refer to the elements of the tuple as S_M, R_M, and V_M.

A model is called *finite* if S is finite. We call model N a *submodel* of M if $S_N \subseteq S_M$, and R_N and V_N are restrictions of R_M and V_M to S_N. We will also write $M_s^X = (S^X, R^X, V^X)$, where $X \subseteq S$, $s \in X$, $S^X = X$, $R^X(a) = R(a) \cap (X \times X)$ for all $a \in A$, and $V^X(p) = V(p) \cap X$ for all $p \in P$.

It is assumed that for group announcements, agents know the formulas they announce. In the following, we write $\mathcal{EL}^G = \{\bigwedge_{i\in G} \Box_i\psi_i \mid \text{for all } i \in G, \psi_i \in \mathcal{EL}\}$ (with typical elements ψ_G) to denote the set of all possible announcements by agents from group G.

Definition 2.5. Let $M_s = (S, R, V)$ be a model, $p \in P$, $G \subseteq A$, and $\varphi, \psi \in \mathcal{APALC} \cup \mathcal{GALC}$.

$$\begin{array}{lll}
M_s \models p & \text{iff} & s \in V(p)\\
M_s \models \neg\varphi & \text{iff} & M_s \not\models \varphi\\
M_s \models \varphi \wedge \psi & \text{iff} & M_s \models \varphi \text{ and } M_s \models \psi\\
M_s \models \Box_a\varphi & \text{iff} & M_t \models \varphi \text{ for all } t \in S \text{ such that } R(a)(s,t)\\
M_s \models \blacksquare_G\varphi & \text{iff} & \forall n \in \mathbb{N} : M_s \models \Box_G^n\varphi\\
M_s \models [\psi]\varphi & \text{iff} & M_s \models \psi \text{ implies } M_s^\psi \models \varphi\\
M_s \models [!]\varphi & \text{iff} & M_s \models [\psi]\varphi \text{ for all } \psi \in \mathcal{EL}\\
M_s \models [G]\varphi & \text{iff} & M_s \models [\psi_G]\varphi \text{ for all } \psi_G \in \mathcal{EL}^G
\end{array}$$

where $M_s^\psi = (S^\psi, R^\psi, V^\psi)$ with $S^\psi = \{s \in S \mid M_s \models \psi\}$, $R^\psi(a)$ is the restriction of $R(a)$ to S^ψ for all $a \in A$, and $V^\psi(p) = V(p) \cap S^\psi$ for all $p \in P$.

It is immediate from the semantics that common knowledge of a group consisting of a single agent is equivalent to the knowledge of that agent: $\blacksquare_{\{a\}}\varphi \leftrightarrow \Box_a\varphi$.

In what follows, we will sometimes say *φ-state* to refer to a state in a given model that satisfies φ.

As discussed in the introduction, we now define the semantics of alternative variants of APAL and GAL extended with common knowledge, where the quantification also ranges over common knowledge formulas.

Let $\mathcal{ELC}^G = \{\bigwedge_{i \in G} \Box_i \psi_i \mid \text{for all } i \in G, \psi_i \in \mathcal{ELC}\}$. Intuitively, $\mathcal{ELC}^G$ is the set of possible group announcements by agents from G that may include common knowledge.

Definition 2.6. Let $M_s = (S, R, V)$ be a model, $p \in P$, $G \subseteq A$, and $\varphi, \psi \in \mathcal{APALC}^X \cup \mathcal{GALC}^X$. The semantics of APALC^X and GALC^X is as in Definition 2.5 with the following modification:

$$\begin{array}{lll} M_s \models [!]^X\varphi & \text{iff} & M_s \models [\psi]\varphi \text{ for all } \psi \in \mathcal{ELC} \\ M_s \models [G]^X\varphi & \text{iff} & M_s \models [\psi_G]\varphi \text{ for all } \psi_G \in \mathcal{ELC}^G \end{array}$$

Note that in a language with *both* types of operators, $[!]^X\varphi \to [!]\varphi$ and $[G]^X\varphi \to [G]\varphi$ would be true in every model.

Definition 2.7. We call formula φ *valid* if and only if for all M_s it holds that $M_s \models \varphi$.

For convenience, let us also provide the semantics for diamonds:

$$\begin{array}{lll} M_s \models \Diamond_a\varphi & \text{iff} & M_t \models \varphi \text{ for some } t \in S \text{ such that } R(a)(s,t) \\ M_s \models \blacklozenge_G\varphi & \text{iff} & \exists n \in \mathbb{N} : M_s \models \Diamond_G^n\varphi \\ M_s \models \langle\psi\rangle\varphi & \text{iff} & M_s \models \psi \text{ and } M_s^\psi \models \varphi \\ M_s \models \langle!\rangle\varphi & \text{iff} & M_s \models \langle\psi\rangle\varphi \text{ for some } \psi \in \mathcal{EL} \\ M_s \models \langle G\rangle\varphi & \text{iff} & M_s \models [\psi_G]\varphi \text{ for some } \psi_G \in \mathcal{EL}^G \\ M_s \models \langle!\rangle^X\varphi & \text{iff} & M_s \models \langle\psi\rangle\varphi \text{ for some } \psi \in \mathcal{ELC} \\ M_s \models \langle G\rangle^X\varphi & \text{iff} & M_s \models [\psi_G]\varphi \text{ for some } \psi_G \in \mathcal{ELC}^G \end{array}$$

It is common in the literature [24, Chapter 2] to define common knowledge of group G via reflexive transitive closure of $\bigcup_{a \in G} R(a)$. We denote such a relation by $R(G)$. The corresponding definition of the semantics then looks like the following:

$$M_s \models \blacksquare_G\varphi \quad \text{iff} \quad M_t \models \varphi \text{ for all } t \in S \text{ such that } R(G)(s,t)$$

Both definitions of common knowledge, via $\Box_G^n\varphi$ for all $n \in \mathbb{N}$ and via $R(G)$, are equivalent to each other, and we will use them interchangeably.

2.2 Bisimulation and Expressivity

We will also use several notions of bisimulation.

Definition 2.8. Let Q be a set of propositional variables, and $M = (S_M, R_M, V_M)$ and $N = (S_N, R_N, V_N)$ be models. We say that M and N are *Q-bisimilar* (denoted $M \leftrightarrows_Q N$) if there is a non-empty relation $B \subseteq S_M \times S_N$, called *Q-bisimulation*, such that for all $B(s,t)$, the following conditions are satisfied:

Atoms for all $p \in Q$: $s \in V_M(p)$ if and only if $t \in V_N(p)$,

Forth for all $a \in A$ and $u \in S_M$ such that $R_M(a)(s,u)$, there is a $v \in S_N$ such that $R_N(a)(t,v)$ and $B(u,v)$,

Back for all $a \in A$ and $v \in S_N$ such that $R_N(a)(t,v)$, there is a $u \in S_M$ such that $R_M(a)(s,u)$ and $B(u,v)$.

We say that M_s and N_t are Q-bisimilar and denote this by $M_s \leftrightarrows_Q N_t$ if there is a Q-bisimulation linking states s and t. Also, we omit subscripts Q if $Q = P$.

Theorem 1. Given M_s and N_t, if $M_s \leftrightarrows N_t$, then for all $\varphi \in \mathcal{APALC} \cup \mathcal{APALC}^X \cup \mathcal{GALC} \cup \mathcal{GALC}^X$ we have that $M_s \models \varphi$ if and only if $N_t \models \varphi$.

Proof. The proof is by induction on φ. Propositional, boolean, and epistemic cases are as usual. The case of common knowledge is proven in [20, Theorem 8.35], and the case of public announcements follows from the corresponding result for action models [20, Theorem 6.21]. Finally, the cases of arbitrary and group announcements follow from the induction hypothesis and the fact that public announcements preserve bisimilarity. $\square$

For the case of Q-bisimulation where $Q \subset P$, Theorem 1 holds only for $\varphi \in \mathcal{PALC}$ that include propositional variables only from Q. The reason the result in this case cannot be extended to a language with quantified announcements is that the quantification is *implicit*, and hence can use propositional variables outside of Q.

Definition 2.9. Let $M = (S, R, V)$ be a model. The *bisimulation contraction* of M is the model $\|M\| = (\|S\|, \|R\|, \|V\|)$, where $\|S\| = \{[s] \mid s \in S\}$ and $[s] = \{t \in S \mid M_s \leftrightarrows M_t\}$, $\|R\|(a)([s],[t])$ if and only if $\exists s' \in [s], \exists t' \in [t]$ such that $R(a)(s',t')$ in M, and $[s] \in \|V\|(p)$ if and only if $\exists s' \in [s]$ such that $s' \in V(p)$.

Intuitively, the bisimulation contraction is the most compact representation of a model. It is a classic result that $M_s \leftrightarrows \|M\|_{[s]}$ [31].

Definition 2.10. Let $n \in \mathbb{N}$, and $M = (S_M, R_M, V_M)$ and $N = (S_N, R_N, V_N)$ be models. We say that M_s and N_t are *n-bisimilar* (denoted $M_s \leftrightarrows^n N_t$) if there exists a sequence of binary relations $B_n \subseteq \ldots \subseteq B_0$ such that

Relation $B_n(s,t)$,

Atoms if $B_0(s',t')$, then for all $p \in P$: $s' \in V_M(p)$ if and only if $t' \in V_N(p)$,

Forth if $B_{i+1}(s',t')$, then for all $a \in A$ and $u \in S_M$ such that $R_M(a)(s',u)$, there is a $v \in S_N$ such that $R_N(a)(t',v)$ and $B_i(u,v)$,

Back if $B_{i+1}(s', t')$, then for all $a \in A$ and $v \in S_N$ such that $R_N(a)(t', v)$, there is a $u \in S_M$ such that $R_M(a)(s', u)$ and $B_i(u, v)$.

It is a standard result that $M_s \leftrightarrows^n N_t$ implies $M_s \models \varphi$ if and only if $N_t \models \varphi$ for $\varphi \in \mathcal{EL}$ with modal depth less or equal n (see, e.g, [31]). This does not hold if φ contains either a common knowledge modality or a quantified announcement. In the first case, common knowledge can access a state on an arbitrarily long distance from the origin. In the second case, quantified announcements are not restricted by any modal depth.

If n-bisimulation between M_s and N_t is restricted to $Q \subset P$, then we will write $M_s \leftrightarrows^n_Q N_t$, and say that M_s and N_t are *Q-n-bisimilar*.

Definition 2.11. Let $\varphi \in \mathcal{L}_1$ and $\psi \in \mathcal{L}_2$. We say that φ and ψ are *equivalent*, if for all M_s: $M_s \models \varphi$ if and only if $M_s \models \psi$.

Definition 2.12. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two languages. If for every $\varphi \in \mathcal{L}_1$ there is an equivalent $\psi \in \mathcal{L}_2$, we write $\mathcal{L}_1 \leqslant \mathcal{L}_2$ and say that $\mathcal{L}_2$ is *at least as expressive as* $\mathcal{L}_1$. We write $\mathcal{L}_1 < \mathcal{L}_2$ if and only if $\mathcal{L}_1 \leqslant \mathcal{L}_2$ and $\mathcal{L}_2 \nleqslant \mathcal{L}_1$, and we say that $\mathcal{L}_2$ is *strictly more expressive than* $\mathcal{L}_1$. If $\mathcal{L}_1 \nleqslant \mathcal{L}_2$ and $\mathcal{L}_2 \nleqslant \mathcal{L}_1$, we say that $\mathcal{L}_1$ and $\mathcal{L}_2$ are *incomparable*.

3 Sharing common knowledge

As one of the main purposes of communication is sharing information, in the context of quantified announcements it is quite natural to ask whether a set of agents can make some fact common knowledge among themselves and other agents. We now state a number of observations for GALC and APALC, but they do in fact all hold for APALC^X and GALC^X as well.

We start with showing that, in general, if a group of agents jointly knows φ, then it is not always the case that they can share this knowledge with another group in such a way that φ becomes commonly known among the members of the other group. A counter-example is the well known Moore sentence (see the extended discussion in the setting of EL in [35]): p is true and agent a does not know this.

Proposition 1. There is a φ such that $\Box_G \varphi \to \langle G \rangle \blacksquare_H \varphi$ and $\Box_G \varphi \to \langle ! \rangle \blacksquare_H \varphi$ are not valid for $G \neq H$.

Proof. Let $G = \{b\}$, $H = \{a\}$, and $\varphi := p \wedge \neg \Box_a p$. Consider model M^1_s in Figure 1, where agent b's relation is the identity. It is clear that $M^1_s \models \varphi$, and hence $M^1_s \models \Box_b \varphi$. Moreover, there are only two possible ways to update M^1_s: leave the model as it is, and remove state t (thus resulting in model M^2_s from the same Figure 1). It is straightforward to verify that $M^1_s \not\models \Box_a \varphi$ and $M^2_s \not\models \Box_a \varphi$, thus resulting in $M^1_s \not\models \langle \{b\} \rangle \Box_a \varphi$ and $M^1_s \not\models \langle ! \rangle \Box_a \varphi$. $\Box$

It is also the case that it is not always possible to share common knowledge of one group with some other group.

Proposition 2. There is a φ such that $\blacksquare_G\varphi \to \langle G\rangle\blacksquare_H\varphi$ and $\blacksquare_G\varphi \to \langle !\rangle\blacksquare_H\varphi$ are not valid for $G \neq H$.

Proof. Follows from Proposition 1 and $\blacksquare_{\{b\}}\varphi \leftrightarrow \Box_b\varphi$. $\Box$

We have the next proposition as a corollary with $\psi := p\vee\neg p$. Informally, the proposition says that it is not always possible for two groups of agents to exchange their common knowledge with one another.

Proposition 3. There are φ and ψ such that $\blacksquare_G\varphi \wedge \blacksquare_H\psi \to \langle G\cup H\rangle\blacksquare_{G\cup H}(\varphi\wedge\psi)$ and $\blacksquare_G\varphi \wedge \blacksquare_H\psi \to \langle !\rangle\blacksquare_{G\cup H}(\varphi\wedge\psi)$ are not valid.

Proof. Let $G = \{b\}$, $H = \{a\}$, $\varphi := p \wedge \neg\Box_a p$, and $\psi := p \vee \neg p$. The proof is similar to the proof of Proposition 1 with $\blacksquare_{\{a\}}\varphi \leftrightarrow \Box_a\varphi$ and $\blacksquare_{\{b\}}\varphi \leftrightarrow \Box_b\varphi$. $\Box$

Interestingly, it is not always possible to make group knowledge common even among the members of the group.

Proposition 4. There is a φ such that $\Box_G\varphi \to \langle G\rangle\blacksquare_G\varphi$ and $\Box_G\varphi \to \langle !\rangle\blacksquare_G\varphi$ are not valid.

Proof. Let $G = \{a, b\}$ and $\varphi := \Diamond_a(\Diamond_a p \wedge \Diamond_b\Box_a\neg p)$, and consider model M_s in Figure 1. Formula φ holds in states s and t of M.

$$
\begin{tikzcd}
 & s & t & u \\
M: & p \arrow[r, dash, dashed] & \neg p \arrow[r, dash] & \neg p \\
 & s & t & \\
M^1: & p \arrow[r, dash, dashed] & \neg p & \\
 & s & & \\
M^2: & p & &
\end{tikzcd}
$$

Figure 1: Model M and some of its submodels. Relation for agent a is depicted by dashed lines and b's relation is shown by solid lines.

It is easy to verify that $M_s \models \Box_{\{a,b\}}\varphi$, and at the same time $M_s \not\models \blacksquare_{\{a,b\}}\varphi$ (the rightmost state of the model, u, does not satisfy φ). Now let us consider all updates of M_s depicted in Figure 1. The reader can check that none of the updates satisfy $\blacksquare_{\{a,b\}}\varphi$. Hence, $M_s \not\models \langle G\rangle\blacksquare_G\varphi$ and $M_s \not\models \langle !\rangle\blacksquare_G\varphi$. $\Box$

All the negative results of this section should not come as a surprise. Target formulas in our proof contained modalities expressing that an agent *does not know* something. Achieving an epistemic goal that also requires someone to remain ignorant of some fact is quite tricky in the setting of public communication. Indeed, formulas with negated knowledge modalities are unstable in the sense that providing additional public information may make them false.

However, for many applications in AI and multi-agent systems, having a stable, easily verifiable epistemic goal is desirable. Examples of such applications include reading a blockchain ledger and alternating bit protocol. See more on this in [19]. It is known that formulas of the *positive* fragment remain true after public communication [23], and below we show that for positive formulas our intuitions regarding sharing common knowledge are indeed true.

Definition 3.1. The *positive fragment of epistemic logic with common knowledge* $\mathcal{ELC}^+$ is defined by the following BNF:

$$\mathcal{ELC}^+ \ \ni\ \varphi^+ ::= p \mid \neg p \mid (\varphi^+ \wedge \varphi^+) \mid (\varphi^+ \vee \varphi^+) \mid \Box_a \varphi^+ \mid \blacksquare_G \varphi^+$$

where $p \in P$, $a \in A$, and $G \subseteq A$. We call $\mathcal{ELC}^+$ without $\blacksquare_G \varphi^+$ the *positive fragment of epistemic logic* $\mathcal{EL}^+$.

The distinctive feature of positive formulas is that they are preserved under submodels, i.e. if φ^+ holds in a model, then φ^+ also holds in all submodels of the model in the same state of evaluation. In particular, this fact implies the following result.

Lemma 1. Let $\varphi^+ \in \mathcal{ELC}^+$, then $[\varphi^+]\blacksquare_G \varphi^+$ is valid for any $G \subseteq A$.

Proof. The proof for the case of common knowledge of the whole set of agents $\blacksquare_A \varphi^+$ can be found in [23]. It is easily adapted to any $G \subseteq A$. $\Box$

Proposition 5. All of the following are valid for any $\phi^+, \psi^+ \in \mathcal{ELC}^+$:

1. $\Box_G \varphi^+ \to \langle G \rangle \blacksquare_H \varphi^+$
2. $\blacksquare_G \varphi^+ \to \langle G \rangle \blacksquare_H \varphi^+$
3. $\blacksquare_G \varphi^+ \wedge \blacksquare_H \psi^+ \to \langle G \cup H \rangle \blacksquare_{G \cup H} (\varphi^+ \wedge \psi^+)$
4. $\Box_G \varphi^+ \to \langle G \rangle \blacksquare_G \varphi^+$
5. $\Box_G \varphi^+ \to \langle ! \rangle \blacksquare_H \varphi^+$
6. $\blacksquare_G \varphi^+ \to \langle ! \rangle \blacksquare_H \varphi^+$
7. $\blacksquare_G \varphi^+ \wedge \blacksquare_H \psi^+ \to \langle ! \rangle \blacksquare_{G \cup H} (\varphi^+ \wedge \psi^+)$
8. $\Box_G \varphi^+ \to \langle ! \rangle \blacksquare_G \varphi^+$

Proof. We outline the general idea for proving all of the statements. First, note that formula $\Box_G \varphi^+$ is already in a form of a group announcement by G (also, for the case of common knowledge we have that $\blacksquare_G \varphi^+ \to \Box_G \varphi^+$). Moreover, $\Box_G \varphi^+$ is positive and holds in the current state of a model. These two facts, in conjunction with Lemma 1, yield $\Box_G \varphi^+ \wedge [\Box_G \varphi^+] \blacksquare_G \Box_G \varphi^+$. The latter is equivalent to $\langle \Box_G \varphi^+ \rangle \blacksquare_G \Box_G \varphi^+$ due to the validity of $\psi \wedge [\psi]\varphi \leftrightarrow \langle \psi \rangle \varphi$. Noting that $\blacksquare_G \Box_G \varphi^+ \to \blacksquare_G \varphi^+$ is valid, we have that $\langle \Box_G \varphi^+ \rangle \blacksquare_G \Box_G \varphi^+$ implies $\langle \Box_G \varphi^+ \rangle \blacksquare_G \varphi^+$. The latter is equivalent to $\langle G \rangle \blacksquare_G \varphi^+$ by the semantics. Finally, $\langle ! \rangle \blacksquare_G \varphi^+$ is implied by $\langle G \rangle \blacksquare_G \varphi^+$. $\Box$

Again, all the results above hold for APALC^X and GALC^X as well, substituting the corresponding modalities.

4 Expressivity

In the previous section we did not find any explicit distinction between GALC and GALC^X, since all the results were true for both. An interesting question, then, is whether there is any difference in expressive power between GALC and GALC^X, and APALC and APALC^X. In this section we show that not only are they different but, perhaps even more surprisingly, they are in fact incomparable. We also situate these languages within a wider context of logics based on EL.

We note that the real difference in expressivity between logics of quantified announcements with common knowledge and their extended versions is only visible on *infinite* models. Indeed, as we claim in the next theorem, both pairs $\mathcal{APALC}$ and $\mathcal{APALC}^X$, and $\mathcal{GALC}$ and $\mathcal{GALC}^X$, are equally expressive on *finite* models.

Theorem 2. Let $M_s = (S, R, V)$ be a finite model. Then $M_s \models \langle ! \rangle \varphi$ if and only if $M_s \models \langle ! \rangle^X \varphi$, and $M_s \models \langle G \rangle \varphi$ if and only if $M_s \models \langle G \rangle^X \varphi$.

Proof. Left-to-right directions of both statements are immediate. Now assume that for some finite M_s, we have $M_s \models \langle ! \rangle^X \varphi$. Without loss of generality, we also assume that M_s is bisimulation contracted. By the definition of semantics, we have that $M_s \models \langle \psi \rangle \varphi$ for some $\psi \in \mathcal{ELC}$. Since M_s is finite, S^ψ is also finite. It is known that in a finite model each state can be uniquely characterised (up to bisimulation) by a distinguishing formula from $\mathcal{EL}$, i.e. a formula that is true only in this state (and all bisimilar states) [15, 6]. Hence, we can construct an announcement that will have the same effect as ψ: $\chi := \bigvee_{t \in S^\psi} \delta_t$, where δ_t's are distinguishing formulas of states t in model M. Since $S^\psi = S^\chi$, we have that $M_s^\psi \models \varphi$ if and only if $M_s^\chi \models \varphi$, which implies $M_s \models \langle \chi \rangle \varphi$ and $M_s \models \langle ! \rangle \varphi$. The same approach can be used for group announcements. $\Box$

4.1 Logics of quantified announcements with common knowledge relative to other logics

Before venturing into the problem of relative expressivity of $\mathcal{APALC}$, $\mathcal{APALC}^X$, $\mathcal{GALC}$, and $\mathcal{GALC}^X$, we compare the aforementioned logics to other logics discussed in the paper. We hope that this section will strengthen the reader's intuitions about quantified announcements, and highlight the crucial role of Q-bisimulation in the coming proofs.

First of all, it is known from the literature that $\mathcal{EL} < \mathcal{ELC} < \mathcal{PALC}$ [12]. Now, we turn our attention to the logics with quantification over public announcements.

Theorem 3. $\mathcal{PALC} < \mathcal{GALC}$, $\mathcal{PALC} < \mathcal{GALC}^X$, $\mathcal{PALC} < \mathcal{APALC}$, and $\mathcal{PALC} < \mathcal{APALC}^X$.

Proof. The proof is quite similar to those for $\mathcal{PAL} < \mathcal{GAL}$ [2, Theorem 19] and $\mathcal{PAL} < \mathcal{APAL}$ [9, Proposition 3.13]. We, however, provide some details here for completeness' sake.

First, we show that $\mathcal{PALC} < \mathcal{GALC}$ (the proof $\mathcal{PALC} < \mathcal{GALC}^X$ is similar). That $\mathcal{PALC} \leqslant \mathcal{GALC}$ follows trivially from the fact that $\mathcal{PALC} \subset \mathcal{GALC}$. To see that $\mathcal{GALC} \not\leqslant$

$\mathcal{PALC}$, consider formula $\langle b \rangle \Box_a p$, and assume towards a contradiction that there is an equivalent formula $\psi \in \mathcal{PALC}$. Since ψ has a finite number of symbols, there must be a propositional variable $q \in P$ that does not occur in ψ. Now consider models M_s and N_s depicted in Figure 2. It is clear that the two models are $P \setminus \{q\}$-bisimilar, and thus

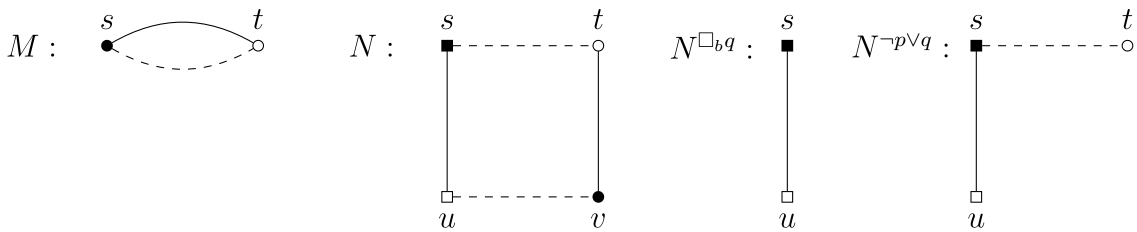

Figure 2: Models M, N, and $N^{\Box_b q}$. Relation for agent a is depicted by dashed lines and b's relation is shown by solid lines. Propositional variable p is true in black states, and propositional variable q is true in square states.

they cannot be distinguished by ψ. On the other hand, we have that $M_s \not\models \langle b \rangle \Box_a p$, since all $\Box_b \varphi$ that are true in s will also be true in t. This is not the case for model N_s. Indeed, announcement of $\Box_b q$ results in $N_s^{\Box_b q}$ for which it holds that $N_s^{\Box_b q} \models \Box_a p$. Hence, $N_s \models \langle b \rangle \Box_a p$, and we have $\mathcal{GALC} \not\leqslant \mathcal{PALC}$.

Now we argue that $\mathcal{PALC} < \mathcal{APALC}$ (again, the proof $\mathcal{PALC} < \mathcal{APALC}^X$ is similar). The fact that $\mathcal{PALC} \subset \mathcal{APALC}$ entails that $\mathcal{PALC} \leqslant \mathcal{APALC}$. Next, we consider formula $\langle ! \rangle (\neg \Box_a p \wedge \Diamond_b \Box_a \neg p)$ of $\mathcal{APALC}$, and assume towards a contradiction that there is an equivalent $\psi \in \mathcal{PALC}$ that does not contain atom q. Similarly to the previous case, we see that ψ cannot distinguish M_s and N_s. To argue that $M_s \not\models \langle ! \rangle (\neg \Box_a p \wedge \Diamond_b \Box_a \neg p)$ it is enough to notice that the only two model updates available in M_s are the trivial one (the model remains intact), and the one that removes state t. In both cases, formula $\neg \Box_a p \wedge \Diamond_b \Box_a \neg p$ is not satisfied. Contrary to that, $N_s \models \langle ! \rangle (\neg \Box_a p \wedge \Diamond_b \Box_a \neg p)$. Indeed, consider announcement of formula $\neg p \vee q$ that results in model $N_s^{\neg p \vee q}$. It is easy to check that $N_s^{\neg p \vee q} \models \neg \Box_a p \wedge \Diamond_b \Box_a \neg p$, thus implying $N_s \models \langle ! \rangle (\neg \Box_a p \wedge \Diamond_b \Box_a \neg p)$ by the semantics, and $\mathcal{APALC} \not\leqslant \mathcal{PALC}$. $\Box$

In the proof of the next theorem we exploit the fact that a given formula with common knowledge modality can reach states on arbitrary distance from a given state. In other words, while modal depth of a given formula is some specific number n, presence of common knowledge modality forces us to consider states on distances greater than n. This is something we will have to take care of in proofs of Section 4.3.

Theorem 4. Both $\mathcal{ELC}$ and $\mathcal{GAL}$, and $\mathcal{ELC}$ and $\mathcal{APAL}$, are incomparable.

Proof. In one direction, the proof is exactly like the proof of Theorem 3.

For the other direction, i.e. to see that $\mathcal{ELC} \not\leqslant \mathcal{GAL}$, consider $\blacksquare_{\{a,b\}} \neg p \in \mathcal{ELC}$ and assume that there is an equivalent $\psi \in \mathcal{GAL}$. As ψ is finite, it must have some finite modal depth n.

Now, let us consider models M and N depicted in Figure 3. Lengths of the models are $n+1$. It is easy to see that $M_s \not\models \blacksquare_{\{a,b\}}\neg p$ and $N_t \models \blacksquare_{\{a,b\}}\neg p$

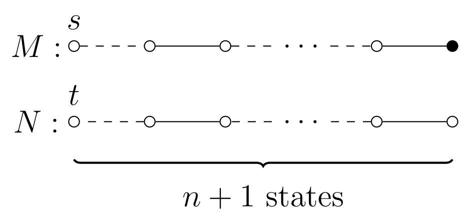

Figure 3: Models M and N. Relation for agent a is depicted by dashed lines and b's relation is shown by solid lines. Propositional variable p is true in the black state.

To show that $M_s \models \psi$ if and only if $N_t \models \psi$, we use the induction on the size of ψ. Since the models are n-bisimilar, no $\mathcal{EL}$ formula of modal depth n can distinguish M_s and N_t.

Case $\psi := [\chi]\tau$ and for some $m < n$, u and v, M_u and N_v are $(n-m)$-bisimilar, where m is a current number of a step in the induction, and u and v are states, where we may have ended up (e.g. after epistemic cases). There are two possible cases. First, update of M with χ preserves the path to the black state. Then, however, τ has a modal depth of at most $(n-m)-1$, while M_u^χ and N_v^χ are $(n-m)-1$-bisimilar. Second, update with χ may not preserve the path to the black state. In this case the two models become bisimilar, and thus cannot be distinguished by any τ.

Cases $\psi := [G]\chi$ and $\psi := \langle ! \rangle \chi$ are like the previous one noting that in the first case we quantify over $\mathcal{EL}^G$ and in the second case we quantify over $\mathcal{EL}$. $\square$

We have the following two theorems as corollaries, noting that $\blacksquare_{\{a,b\}}\neg p$ is also a formula of $\mathcal{PALC}$, $\mathcal{APALC}$, $\mathcal{GALC}$, $\mathcal{APALC}^X$, and $\mathcal{GALC}^X$.

Theorem 5. Both pairs $\mathcal{PALC}$ and $\mathcal{GAL}$, and $\mathcal{PALC}$ and $\mathcal{APAL}$, are incomparable.

Theorem 6. $\mathcal{GAL} < \mathcal{GALC}$, $\mathcal{GAL} < \mathcal{GALC}^X$, $\mathcal{APAL} < \mathcal{APALC}$, and $\mathcal{APAL} < \mathcal{APALC}^X$.

4.2 Formula games

One of the classic techniques for comparing expressive power of modal languages is by using games over models [20, Chapter 8]. Such games are usually played between two players, one of which tries to show that the two models are the same, and another one tries to demonstrate that the models are different. Moves in a game are determined by a given formula of a logic, and the number of moves by either player is bounded by the modal depth of the formula.

Formula games for GAL and coalition announcement logic [3, 26] were originally introduced in [25] (see also [26, Chapter 7] for details and examples). Here we introduce formula games for logics of quantified announcements with common knowledge considered in the paper.

Definition 4.1. The set of formulas in *negation normal form* $\mathcal{NNF}$ is defined by the following BNF:

$$\varphi ::= \begin{array}{l} \top \mid \varphi \wedge \varphi \mid \Box_a \varphi \mid \blacksquare_G \varphi \mid [!]\varphi \mid [!]^X \varphi \mid [G]\varphi \mid [G]^X \varphi \\ \mid \ \bot \mid p \mid \neg p \mid \varphi \vee \varphi \mid \Diamond_a \varphi \mid \blacklozenge_G \varphi \mid [\varphi]\varphi \mid \langle ! \rangle \varphi \mid \langle ! \rangle^X \varphi \mid \langle G \rangle \varphi \mid \langle G \rangle^X \varphi \end{array},$$

where $p \in P$ and $G \subseteq A$. If for formula $\varphi \in \mathcal{NNF}$ the outermost operator or the main connective are from the top line, then we say that φ is in *universal negation normal form* $\mathcal{UNNF}$; and if the outermost operator or the main connective are from the line below, then φ is in *existential negation normal norm* $\mathcal{ENNF}$. We would also like to point out the absence of clause $\langle \varphi \rangle \varphi$ in the BNF. As it will become clear later, in Lemma 2, we can do without it.

Lemma 2. Every formula of $\mathcal{APALC}$, $\mathcal{APALC}^X$, $\mathcal{GALC}$, and $\mathcal{GALC}^X$ can be equivalently rewritten to a formula in $\mathcal{NNF}$.

Proof. The proof is a straightforward 'pushing' of negations inside of the scope of operators. We use translation function $t : (\mathcal{APALC} \cup \mathcal{APALC}^X \cup \mathcal{GALC} \cup \mathcal{GALC}^X) \to \mathcal{NNF}$ that is defined as follows:

$$\begin{array}{lcllcl}
t(\neg p) & = & \neg p & t(p) & = & p \\
t(\neg(\varphi \wedge \psi)) & = & t(\neg \varphi) \vee t(\neg \psi) & t(\varphi \wedge \psi) & = & t(\varphi) \wedge t(\psi) \\
t(\neg \Box_a \varphi) & = & \Diamond_a t(\neg \varphi) & t(\Box_a \varphi) & = & \Box_a t(\varphi) \\
t(\neg \blacksquare_G \varphi) & = & \blacklozenge_G t(\neg \varphi) & t(\blacksquare_G \varphi) & = & \blacksquare_G t(\varphi) \\
t(\neg [\psi] \varphi) & = & t(\psi) \wedge t([\psi] \neg \varphi) & t([\psi]\varphi) & = & [t(\psi)] t(\varphi) \\
t(\neg [!] \varphi) & = & \langle ! \rangle t(\neg \varphi) & t([!]\varphi) & = & [!] t(\varphi) \\
t(\neg [!]^X \varphi) & = & \langle ! \rangle^X t(\neg \varphi) & t([!]^X \varphi) & = & [!]^X t(\varphi) \\
t(\neg [G] \varphi) & = & \langle G \rangle t(\neg \varphi) & t([G]\varphi) & = & [G] t(\varphi) \\
t(\neg [G]^X \varphi) & = & \langle G \rangle^X t(\neg \varphi) & t([G]^X \varphi) & = & [G]^X t(\varphi)
\end{array}$$

□

Before we continue with formula games, we introduce a *size* relation that will be helpful in induction proofs of this section.

Definition 4.2. Let φ be a formula. The *size* $s(\varphi)$ of φ is defined inductively as

$$s(p) = 1 \qquad s([\psi]\varphi) = s(\psi) + s(\varphi) + 1$$

$$s(O\varphi) = s(\varphi) + 1 \qquad s(\varphi C \psi) = \max(s(\varphi), s(\psi))$$

In the definition, $O \in \{\neg, \Box_a, \Diamond_a, \blacksquare_G, \blacklozenge_G, [!], \langle ! \rangle, [!]^X, \langle ! \rangle^X, [G], \langle G \rangle, [G]^X, \langle G \rangle^X\}$ and $C \in \{\wedge, \vee\}$. We will write $\varphi <^\forall \psi$ if and only if $\delta_\forall(\varphi) < \delta_\forall(\psi)$ (Definition 2.3), or, otherwise, $\delta_\forall(\varphi) = \delta_\forall(\psi)$ and $s(\varphi) < s(\psi)$.

We will also need an auxiliary lemma that states that a formula and its translation to NNF has the same quantifier depth and size.

Lemma 3. Let $\varphi \in \mathcal{APALC} \cup \mathcal{APALC}^X \cup \mathcal{GALC} \cup \mathcal{GALC}^X$. Then $\delta_\forall(\varphi) = \delta_\forall(t(\varphi))$ and $s(\varphi) = s(t(\varphi))$.

Proof. A proof is straightforward, and we show just one case as an example. Consider $\neg[\psi]\varphi$. Size of this formula, according to Definition 4.2, is $s(\neg[\psi]\varphi) = s([\psi]\varphi) + 1 = s(\psi) + s(\varphi) + 2$. Now, let us take the translation $t(\neg[\psi]\varphi) = t(\psi) \wedge t([\psi]\neg\varphi)$. Size of the translation is $s(t(\psi) \wedge t([\psi]\neg\varphi)) = \max(s(t(\psi)), s(t([\psi]\neg\varphi))) = s(t([\psi]\neg\varphi)) = s([t(\psi)]t(\neg\varphi)) = s(t(\psi)) + s(t(\neg\varphi)) + 1$. Assuming by the induction hypothesis that $s(t(\psi)) = s(\psi)$ and $s(t(\neg\varphi)) = s(\neg\varphi) = s(\varphi) + 1$, we get the desired equality. $\square$

Now we are ready to define formula games that are played between the $\forall$-player (the universal player) and the $\exists$-player (the existential player) over a given model. Types and order of moves are determined by a given formula that the game is constructed for: the universal player moves if a current subformula is in $\mathcal{UNNF}$, and the existential player moves if the current subformula is in $\mathcal{ENNF}$.

Definition 4.3. Let some model M_s and $\varphi \in \mathcal{NNF}$ be given, and suppose that $\mathcal{M}$ is the set of pointed submodels N_t^X of model M_s, where $X \subseteq S$ and $s \in X$. A *formula game for φ over M_s* is a tuple $\mathcal{G}^\varphi_{M_s} = (V_\forall, V_\exists, E, \Delta)$, where

- $V_\forall = \{\ulcorner N_t, \psi \urcorner \mid N_t \in \mathcal{M}, \psi \in \mathcal{UNNF}\} \cup \{\ulcorner N_t, X, \chi, \psi \urcorner \mid N_t \in \mathcal{M}, X \subseteq S, \chi \in \mathcal{NNF}, \psi \in \mathcal{NNF}\}$ is the set of vertices of the $\forall$-player,
- $V_\exists = \{\ulcorner N_t, \psi \urcorner \mid N_t \in \mathcal{M}, \psi \in \mathcal{ENNF}\}$ is the set of vertices of the $\exists$-player,
- $E \subset (V_\forall \cup V_\exists) \times (V_\forall \cup V_\exists)$ is the set of edges, where E is a union of the following sets
  - $\{(\ulcorner N_t, p \urcorner, \ulcorner N_t, \top \urcorner), (\ulcorner N_t, \neg q \urcorner, \ulcorner N_t, \top \urcorner) \mid t \in V(p) \text{ and } t \notin V(q)\}$,
  - $\{(\ulcorner N_t, p \urcorner, \ulcorner N_t, \bot \urcorner), (\ulcorner N_t, \neg q \urcorner, \ulcorner N_t, \bot \urcorner) \mid t \notin V(p) \text{ and } t \in V(q)\}$,
  - $\{(\ulcorner N_t, \psi \wedge \chi \urcorner, \ulcorner N_t, \psi \urcorner), (\ulcorner N_t, \psi \wedge \chi \urcorner, \ulcorner N_t, \chi \urcorner)\}$,
  - $\{(\ulcorner N_t, \psi \vee \chi \urcorner, \ulcorner N_t, \psi \urcorner), (\ulcorner N_t, \psi \vee \chi \urcorner, \ulcorner N_t, \chi \urcorner\}$,
  - $\{(\ulcorner N_t, \Box_a \psi \urcorner, \ulcorner N_u, \psi \urcorner) \mid R(a)(t, u)\}$,
  - $\{(\ulcorner N_t, \Diamond_a \psi \urcorner, \ulcorner N_u, \psi \urcorner) \mid R(a)(t, u)\}$,
  - $\{(\ulcorner N_t, \blacksquare_G \psi \urcorner, \ulcorner N_u, \psi \urcorner) \mid R(G)(t, u)\}$,
  - $\{(\ulcorner N_t, \blacklozenge_G \psi \urcorner, \ulcorner N_u, \psi \urcorner) \mid R(G)(t, u)\}$,
  - $\{(\ulcorner N_t, [\chi]\psi \urcorner, \ulcorner N_t, X, \chi, \psi \urcorner)\}$,
  - $\{(\ulcorner N_t, X, \chi, \psi \urcorner, \ulcorner N_u, \chi \urcorner) \mid u \in X\}$,
  - $\{(\ulcorner N_t, X, \chi, \psi \urcorner, \ulcorner N_u, t(\neg\chi) \urcorner) \mid u \in S \setminus X\}$,
  - $\{(\ulcorner N_t, X, \chi, \psi \urcorner, \ulcorner N_t^X, \psi \urcorner)\}$,
  - $\{(\ulcorner N_t, [!]\psi \urcorner, \ulcorner N_t, [t(\chi)]\psi \urcorner) \mid \chi \in \mathcal{EL}\}$,
  - $\{(\ulcorner N_t, \langle ! \rangle \psi \urcorner, \ulcorner N_t, t(\chi) \wedge [t(\chi)]\psi \urcorner) \mid \chi \in \mathcal{EL}\}$,

- $\{(\ulcorner N_t, [!]^X\psi\urcorner, \ulcorner N_t, [t(\chi)]\psi\urcorner) \mid \chi \in \mathcal{ELC}\}$,
- $\{(\ulcorner N_t, \langle ! \rangle^X\psi\urcorner, \ulcorner N_t, t(\chi) \wedge [t(\chi)]\psi\urcorner) \mid \chi \in \mathcal{ELC}\}$,
- $\{(\ulcorner N_t, [G]\psi\urcorner, \ulcorner N_t, [t(\chi_G)]\psi\urcorner) \mid \chi_G \in \mathcal{EL}^G\}$,
- $\{(\ulcorner N_t, \langle G \rangle\psi\urcorner, \ulcorner N_t, t(\chi_G) \wedge [t(\chi_G)]\psi\urcorner) \mid \chi_G \in \mathcal{EL}^G\}$,
- $\{(\ulcorner N_t, [G]^X\psi\urcorner, \ulcorner N_t, [t(\chi_G)]\psi\urcorner) \mid \chi_G \in \mathcal{ELC}^G\}$,
- $\{(\ulcorner N_t, \langle G \rangle^X\psi\urcorner, \ulcorner N_t, t(\chi_G) \wedge [t(\chi_G)]\psi\urcorner) \mid \chi_G \in \mathcal{ELC}^G\}$.

- Δ is the initial vertex $\ulcorner M_s, \varphi\urcorner$.

The game is played between the $\forall$-player and the $\exists$-player, and a *play* consists of a sequence of vertices $\Delta, \Delta_1, \ldots, \Delta_n$. The play is built by the players such that for some edge $(\Delta_m, \Delta_{m+1}) \in E$ if $\Delta_m \in V_\forall$, then the universal player chooses Δ_{m+1}, and if $\Delta_m \in V_\exists$, then the existential player chooses Δ_{m+1}. If either player is unable to move, i.e. they are in a $\top$-vertex or $\bot$-vertex, then they lose the game.

The intuition behind edges of a game is that they show which moves the current player has. For example, if we are in vertex $\ulcorner N_t, \psi \wedge \chi\urcorner$ of a game, then the $\forall$-player can either choose to move to vertex $\ulcorner N_t, \psi\urcorner$ or to vertex $\ulcorner N_t, \chi\urcorner$. If we are in vertex $\ulcorner N_t, \blacklozenge_G\psi\urcorner$ of the game, then the $\exists$-player can choose any state u of N reachable from t via $R(G)$, thus letting the game to carry on in vertex $\ulcorner N_u, \psi\urcorner$.

Of special interest are moves that correspond to public announcements and quantifiers. From vertex $\ulcorner N_t, [\chi]\psi\urcorner$ the existential player can move to a vertex $\ulcorner N_t, X, \chi, \psi\urcorner$, where X is a subset of S_N. From this position, the universal player can challenge the choice of the existential player in three different ways. First, she can check whether $X \subseteq \{u \in S_N \mid N_u \models \chi\}$, i.e. whether all states in the chosen subset satisfy χ. Second, the universal player can check whether $S \setminus X \subseteq \{u \in S_N \mid N_u \models t(\neg\chi)\}$, i.e. whether all states outside of X are $\neg\chi$-states. The third option is to continue the game in a submodel N_t^X with the formula ψ. All these choices of the universal player correspond to the semantics of public announcements.

Finally, the game positions with quantified announcements also follow the semantics. For example, in vertex $\ulcorner N_t, \langle G \rangle\psi\urcorner$ of the game, the existential player can choose *any* formula $\chi_G \in \mathcal{EL}^G$ thus making a move to vertex $\ulcorner N_t, t(\chi_G) \wedge [t(\chi_G)]\psi\urcorner$, where the universal player can either check that the chosen formula is indeed true, or let the $\exists$-player to carry on with announcement of the chosen formula.

In the next proposition we show that all plays of formula games are finite.

Proposition 6. Given formula $\varphi \in \mathcal{NNF}$, model M_s, and a game $\mathcal{G}_{M_s}^{\varphi}$, every play of the game is finite.

Proof. The proof is by structural induction on φ.

Base Case: in the case of a propositional variable there is exactly one step in a play of the game.

Induction Hypothesis (IH): for all pointed submodels N_t of M and for all ψ such that $\psi <^{\forall} \varphi$ (Definition 4.2), plays of the game are finite.

The propositional and epistemic cases are straightforward, so we omit them. Also note that it means that plays for epistemic formulas are finite.

Case $\ulcorner N_t, [\chi]\psi \urcorner$: in this position of the game the existential player chooses a subset X of the set of states S_N of the given model. Such a choice leads to one of the vertices $\ulcorner N_t, X, \chi, \psi \urcorner$. There are three possible choices of the $\forall$-player from this vertex: $\ulcorner N_t, \chi \urcorner$, $\ulcorner N_u, t(\neg\chi) \urcorner$, and $\ulcorner N_t^X, \psi \urcorner$. Observe that $\chi <^{\forall} [\chi]\psi$ and $\psi <^{\forall} [\chi]\psi$, and thus plays from $\ulcorner N_t, \chi \urcorner$ and $\ulcorner N_t^X, \psi \urcorner$ are finite by the IH. Moreover, by Lemma 3 we have that $\delta_{\forall}(t(\neg\chi)) = \delta_{\forall}(\neg\chi)$ and $s(t(\neg\chi)) = s(\neg\chi)$. It holds that $t(\neg\chi) <^{\forall} [\chi]\psi$, and thus plays from $\ulcorner N_u, t(\neg\chi) \urcorner$ are finite by the IH.

Case $\ulcorner N_t, [!]\psi \urcorner$: there is just one step from this vertex to some $\ulcorner N_t,\ [t(\chi)]\psi \urcorner$ such that $\psi \in \mathcal{EL}$. Observe that $[!]\psi <^{\forall} [t(\chi)]\psi$, and thus by the IH, we conclude that the play from this vertex is finite.

Cases $\ulcorner N_t, \langle ! \rangle\psi \urcorner$, $\ulcorner N_t, [!]^X\psi \urcorner$, $\ulcorner N_t, \langle ! \rangle^X\psi \urcorner$, $\ulcorner N_t, [G]\psi \urcorner$, $\ulcorner N_t, \langle G \rangle\psi \urcorner$, $\ulcorner N_t, [G]^X\psi \urcorner$, and $\ulcorner N_t, \langle G \rangle^X\psi \urcorner$ are similar to the previous one. □

In the following proposition we state the relation between a formula being true in the current state of a model, and the existence of the winning strategy for the existential player in the corresponding game.

Proposition 7. The $\exists$-player has a winning strategy in a game $\mathcal{G}_{M_s}^{\varphi}$ if and only if $M_s \models \varphi$.

Proof. From right to left.

Base Case: Assume that $M_s \models p$. Then the corresponding formula game consists only of one $\exists$-step from $\ulcorner M_s, p \urcorner$ to $\ulcorner M_s, \top \urcorner$, and the latter is the winning vertex of the existential player (it it universal player's turn but they cannot move). The same argument holds for $\neg p$.

Induction Hypothesis (IH): Assume that for all pointed submodels N_t of M and all formulas $t(\psi)$ in NNF such that $t(\psi) <^{\forall} \varphi$, if $N_t \models t(\psi)$, then $\ulcorner N_t, t(\psi) \urcorner$ is a winning position for the $\exists$-player.

Propositional and epistemic cases are straightforward.

Case $N_t \models [\psi]\chi$: by the semantics this is equivalent to $N_t \models \neg\psi$ or $N_t^{\psi} \models \chi$. First, assume that $N_t \models \neg\psi$, and consider $X = \{u \in S_N \mid N_u \models \psi\}$ and $Y = S_N \setminus X$, where X can be an empty set. We have that for all $u \in X$: $N_u \models \psi$ and for all $v \in Y$: $N_v \models t(\neg\psi)$. By the IH this implies that $\ulcorner N_u, \psi \urcorner$ and $\ulcorner N_v, t(\neg\psi) \urcorner$ are winning positions for the existential player for all $u \in X$ and $v \in Y$. Hence, $\ulcorner N_t, X, \psi, \chi \urcorner$ is also a winning position for the $\exists$-player that she can choose from $\ulcorner N_t, [\psi]\chi \urcorner$.

If $N_t^{\psi} \models \chi$, then again we consider $X = \{u \in S_N \mid N_u \models \psi\}$ similarly to the case of $N_t \models \neg\psi$. Since $\chi <^{\forall} [\psi]\chi$, then by the IH we have that $\ulcorner N_t^X, \chi \urcorner$ is a winning position for the $\exists$-player. Hence, $\ulcorner N_t, X, \psi, \chi \urcorner$ is a winning position for the $\exists$-player that she can choose from $\ulcorner N_t, [\psi]\chi \urcorner$.

Case $N_t \models \langle ! \rangle\psi$: by the semantics $N_t \models \langle ! \rangle\psi$ is equivalent to $\exists\chi \in \mathcal{EL}$: $N_t \models \langle \chi \rangle\psi$. The latter is equivalent to $N_t \models t(\chi) \wedge t([\chi]\psi)$. Since $t(\chi) \wedge t([\chi]\psi) <^{\forall} \langle ! \rangle\psi$, we can use the

IH to conclude that the $\exists$-player can always choose a step in the game that corresponds to the winning position $\ulcorner N_t, t(\chi) \wedge t([\chi]\psi)\urcorner$. Thus, $\ulcorner N_t, \langle ! \rangle \psi \urcorner$ is also a winning position for the existential player.

Cases for $[!]\psi$, $[!]^X\psi$, $\langle ! \rangle^X \psi$, $[G]\psi$, $\langle G \rangle \psi$, $[G]^X\psi$, and $\langle G \rangle^X \psi$ are similar to the previous one.

From left to right. A similar argument as in the opposite direction for the contraposition: if $M_s \not\models \varphi$, then the $\forall$-player has a winning strategy in game $\mathcal{G}^{\varphi}_{M_s}$. $\square$

To recapitulate, Proposition 7 states that if a formula is true in a model, then the existential player has a winning strategy. Alternatively, if the formula is false in a model, then the universal player has a winning strategy. We will use these facts in the next section, when we will let both players to play their winning strategies against each other.

4.3 APALC and GALC relative to APALC^X and GALC^X

Now we turn to the key question of the relative expressivity of $\mathcal{APALC}$ and $\mathcal{APALC}^X$, and of $\mathcal{GALC}$ and $\mathcal{GALC}^X$. We show in Theorem 7 that there are some properties of models that can be captured by the extended versions of the logics, and cannot be captured by the conservative versions.

We start by presenting two models, M and N in Figure 4 that we will be used in the proof. In both models, there are chains starting from s and t correspondingly of length $n+2$ for each $n \in \mathbb{N}$. Chains end with boxed states. In model N there is also an infinite *vertical* chain starting from state u. Propositional variable p is true in s and t, and q is true in boxed states at the ends of finite chains.

Model M is constructed in such a way that the upper and lower parts of the model (relative to state s) are bisimilar. In particular, $M_{s^u} \leftrightarrows M_{s^l}$, $M_{n^u} \leftrightarrows M_{n^l}$ and $M_{n^u_m} \leftrightarrows M_{n^l_m}$ for all $n, m \in \mathbb{N}$ with $m < n$. This is not the case for model N, where the presence of the infinite vertical chain allows us to distinguish the upper and lower parts of the model. Indeed, take an arbitrary state n^u_m from the upper part. Formula $\neg\blacklozenge_{\{b,c\}} q$ is false in $N_{n^u_m}$, and it is satisfied in N_u (or any other state of the infinite chain).

Next, we show that there are formulas of $\mathcal{APALC}^X$ and $\mathcal{GALC}^X$ that can distinguish M_s and N_t.

Lemma 4. There are formulas $\psi_1 \in \mathcal{APALC}^X$ and $\psi_2 \in \mathcal{GALC}^X$, such that $M_s \not\models \psi_*$ and $N_t \models \psi_*$, where $* \in \{1, 2\}$.

Proof. Let

$$\varphi := p \wedge \Diamond_b(\neg p \wedge \Box_a \Diamond_b p) \wedge \Diamond_b(\Diamond_a \Box_b \neg p \wedge \Box_a(\neg \Diamond_b p \rightarrow \Box_b \Diamond_a \Diamond_b p)),$$

and $\langle ! \rangle^X \varphi \in \mathcal{APALC}^X$. In order to see that $N_t \models \langle ! \rangle^X \varphi$, consider the following announcement:

$$\psi_c := \Box_c((\neg p \rightarrow (\Box_{\{b,c\}} q \vee \Diamond_b p)) \wedge (q \rightarrow \Box_a(\blacklozenge_{\{b,c\}} q \vee \Diamond_b p))).$$

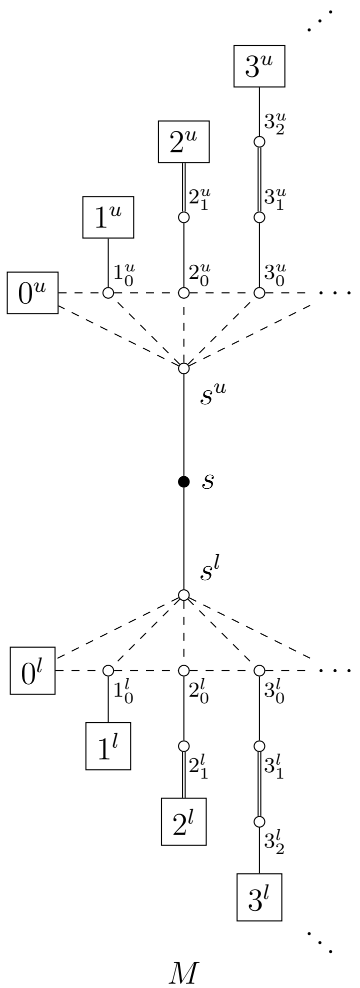

Figure 4: Models M and N. Relation for agent a is depicted by dashed lines, relation b is shown by solid lines, and c's relations are double lines. Propositional variable p is true in black states, and q is true in boxed states at the ends of chains.

Note that we use an announcement with q here, while q does not appear in φ. Also note that this announcement belongs to $\mathcal{ELC}$. In model N, formula $\neg\blacklozenge_{\{b,c\}}q$ is true only in states t, t^u, t^l, and all states of the infinite vertical chain including u.

We now argue that the result of updating N with the announcement is presented in Figure 5.

First, pick any non-zero boxed state, i.e. let $n^* \in \{n^* \mid n \in \mathbb{N}\setminus\{0\}$ and $* \in \{u, l\}\}$. We have that $N_{n^*} \not\models \neg p \to (\Box_{\{b,c\}}q \vee \Diamond_b p)$ as p is true only in the black state and thus cannot be reached by b, and there is always either a b- or c-arrow to a neighbour circle node with $\neg q$. Hence, $N_{n^*} \not\models \psi_c$. Now consider state 0^l: it holds that $N_{0^l} \not\models q \to \Box_a(\blacklozenge_{\{b,c\}}q \vee \Diamond_b p)$ since there is an a-arrow to state u and $N_u \models \neg\blacklozenge_{\{b,c\}}q \wedge \neg\Diamond_b p$. On the other hand, all a-arrows from 0^u lead to states where either $\blacklozenge_{\{b,c\}}q$ or $\Diamond_b p$ hold: each reachable finite chain

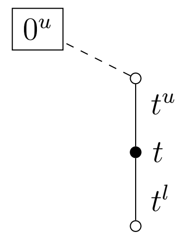

Figure 5: Submodel O of model N.

ends with a q-state, and from state t^u there is a b-arrow to the p-state. It is left to check that $N_{0^u} \models \neg p \to (\Box_{\{b,c\}} q \vee \Diamond_b p)$, and indeed $N_{0^u} \models \Box_{\{b,c\}} q$, and hence $N_{0^u} \models \psi_c$.

Second, pick any circle state apart from t^u and t^l. To see that $N_\circ \not\models \neg p \to (\Box_{\{b,c\}} q \vee \Diamond_b p)$, notice that $N_\circ \models \neg p$, $N_\circ \not\models \Box_{\{b,c\}} q$ (q is false in the current state) and $N_\circ \not\models \Diamond_b p$ (as p is true only in the black state, which is not reachable via b from any white circle state apart from t^u and t^l). So, $N_\circ \not\models \psi_c$. In both t^u and t^l, $\Diamond_b p$ is true and hence the whole formula is true. Finally, we have $N_\bullet \models \psi_c$ vacuously, since $N_\bullet \not\models \neg p$ and $N_\bullet \not\models q$. Thus, the result of updating N with ψ_c is bisimilar to the model O in Figure 5.

It is easy to check that $O_t \models \varphi$. Formula φ is constructed in such a way that it can only be satisfied by model O_t (up to bisimulation). The first conjunct in φ checks the truth of p in the current state. The second conjunct specifies that there is a $\neg p$-state reachable in one b-step that is not a numbered state. Finally, the third conjunct ensures that there is a numbered state reachable in two steps, and no other 'deeper' states are available.

To argue that $M_s \not\models \langle ! \rangle^X \varphi$, we recall that the upper and lower halves of model M (relative to state s) are bisimilar. Now assume towards a contradiction that there is a $\psi \in \mathcal{ELC}$ such that $M_s^\psi \models \varphi$. In particular, we have that $M_s^\psi \models \Diamond_b(\neg p \wedge \Box_a \Diamond_b p)$. By the semantics, this means that there is a state, either s^u or s^l (or both), such that $\Box_a \Diamond_b p$ holds in that state. By the construction of M, the only way $\Box_a \Diamond_b p$ can be satisfied in s^u or s^l is by removing all other a-reachable states. Since $M_{s^u} \leftrightarrows M_{s^l}$, by Theorem 1 we have that $M_{s^u} \models \Box_a \Diamond_b p$ if and only if $M_{s^l} \models \Box_a \Diamond_b p$. But this contradicts the third conjunct of φ. Since ψ was arbitrary, $M_s \not\models \langle ! \rangle^X \varphi$.

Finally, note that the same argument works for $\mathcal{GALC}^X$. Indeed, formula ψ_c belongs to $\mathcal{ELC}^{\{c\}}$, and thus we have that $N_t \models \langle \{c\} \rangle^X \varphi$. The fact that $M_s \not\models \langle \{c\} \rangle^X \varphi$ again follows from the proof of $M_s \not\models \langle ! \rangle^X \varphi$ noting that the choice of ψ was arbitrary. $\Box$

Now we are left to show that models M_s and N_t cannot be distinguished by none of the formulas of $\mathcal{APALC}$ or $\mathcal{GALC}$. For the proof we will use formula games introduced in Section 4.2. First, we will assume towards a contradiction that there is a formula ψ of $\mathcal{APALC}$ or $\mathcal{GALC}$ such that it is true in one model and false in the other. By Proposition 7 this means that the $\exists$-player has a winning strategy in one model, and the $\forall$-player has a winning strategy in the other model.

We will play two games simultaneously, one over M_s and φ, and the other over N_t and φ. In each game each player will play according to their winning strategies. Since games are finite by Proposition 6, we should end up in the situation, where one player has won in one model, and the other player has won in the other model. However, we will use the notion of Q-n-bisimulation to argue that at the final step both players in both models will be in states satisfying the same propositional variables, meaning that one of the winning strategies for one of the players is not winning at all. And this will yield the desired contradiction.

Theorem 7. $\mathcal{APALC}^X \not\leqslant \mathcal{APALC}$ and $\mathcal{GALC}^X \not\leqslant \mathcal{GALC}$.

Proof. In Lemma 4 we have seen that formulas $\langle ! \rangle^X \varphi \in \mathcal{APALC}^X$ and $\langle \{c\} \rangle^X \in \mathcal{GALC}^X$ distinguish models M_s and N_t. Now assume towards a contradiction that there is a $\psi \in \mathcal{APALC} \cup \mathcal{GALC}$ that is equivalent to either $\langle ! \rangle^X \varphi$ or $\langle \{c\} \rangle^X \varphi$ accordingly. Without loss of generality, we also assume that $\psi \in \mathcal{NNF}$. Since ψ has a finite number of symbols, there must be a $q \in P$ such that q does not occur in ψ.

Since ψ is equivalent to $\langle ! \rangle^X \varphi$ or $\langle \{c\} \rangle^X \varphi$, we have that $M_s \not\models \psi$ and $N_t \models \psi$. This means, by Proposition 7, that the $\forall$-player has a winning strategy in $\mathcal{G}^{\psi}_{M_s}$, and the $\exists$-player has a winning strategy in $\mathcal{H}^{\psi}_{N_t}$. Given $n = md(\psi)$, we consider the following relation $B \subseteq S_M \times S_N$:

$$B = \bigcup \left(\begin{array}{l} \{(s,t), (s^u, t^u), (s^l, s^l)\} \\ \{(m^*, m^*) \mid m \in \mathbb{N}, * \in \{u, l\}\} \\ \{(m^*_o, m^*_o) \mid m, o \in \mathbb{N}, o < m, * \in \{u, l\}\} \\ \{((2^n)^l_0, u)\} \cup \{((2^n)^l_k, \omega_k) \mid k < 2^n\} \cup \{((2^n)^l, \omega_k) \mid k \geqslant 2^n\} \end{array} \right).$$

It is clear that B is an $P \setminus \{q\}$-2^n-bisimulation relation between M and N, where each state of one model is in relation to the corresponding state of the other model. As for the infinite chain, we put states on the chain in relation to states from chain $(2^n)^l$ of M in such a way that $M_{(2^n)^l_k} \leftrightarrows^{2^n} N_{\omega_k}$ if $k < 2^n$, and all ω_k with $k \geqslant 2^n$ are put into relation with state $(2^n)^l$ of M. Now we show that after k steps of a game, all the remaining states are still $P \setminus \{q\}$-$(2^n - k)$-bisimilar.

Base Case: Let $\psi = p$ for some $p \in P \setminus \{q\}$. Since all states that are in relation B satisfy the same propositional variables from $P \setminus \{q\}$, we have $P \setminus \{q\}$-0-bisimilarity.

Induction Hypothesis (IH): After k steps of a game, for all states s' and t' from all submodels M' and N', if $B(s', t')$, then $M'_{s'} \leftrightarrows^{2^n-k}_{P \setminus \{q\}} N'_{t'}$.

Cases $\psi = \chi \wedge \tau$ and $\psi = \chi \vee \tau$. In game $\mathcal{G}^{\psi}_{M_s}$, the $\forall$-player makes a move from $\ulcorner M'_{s'} \chi \wedge \tau \urcorner$ to either $\ulcorner M'_{s'}, \chi \urcorner$ or $\ulcorner M'_{s'}, \tau \urcorner$. The universal player makes the same choice (either χ or τ) in game $\mathcal{H}^{\psi}_{N_t}$. Since such a move does not change current states, we have $M'_{s'} \leftrightarrows^{2^n-k}_{P \setminus \{q\}} N'_{t'}$ by the IH, which implies $M'_{s'} \leftrightarrows^{2^n-k-1}_{P \setminus \{q\}} N'_{t'}$. Similarly for $\psi = \chi \vee \tau$ and the $\exists$-player.

Cases $\psi = \Box_a \chi$ and $\psi = \Diamond_a \chi$. In game $\mathcal{G}^{\psi}_{M_s}$, the $\forall$-player makes a move, according to her winning strategy, from $\ulcorner M'_{s'}, \Box_a \chi \urcorner$ to some $\ulcorner M'_{s^*}, \chi \urcorner$ such that $R_M(a)(s', s^*)$. A similar move is made in game $\mathcal{H}^{\psi}_{N_t}$: from $\ulcorner N'_{t'}, \Box_a \chi \urcorner$ to some $\ulcorner N'_{t^*}, \chi \urcorner$ such that $R_N(a)(t', t^*)$ and $M'_{s^*} \leftrightarrows^{2^n-k-1}_{P \setminus \{q\}} N'_{t^*}$. The existence of such a t^* follows from the IH and Definition 2.10.

Note that the way we defined B specifies that if the player made a move in game $\mathcal{H}^{\psi}_{N_t}$ to state u on the infinite chain, then the move will be matched by a move to state $(2^n)^l_0$ in game $\mathcal{G}^{\psi}_{M_s}$, i.e. the first state of chain $(2^n)^l$ in model M. Moves along the infinite chain are matched by moves along chain $(2^n)^l$, and all moves on the infinite chain beyond ω_{2^n} are matched by the player choosing to stay in state $(2^n)^l$, which is the last state of the chain.

The similar reasoning applies to $\psi = \Diamond_a \chi$ and the $\exists$-player.

Cases $\psi = \blacksquare_G \chi$ and $\psi = \blacklozenge_G \chi$. These are cases similar to the previous ones with substituting $R(a)$ by $R(G)$. Again, according to B, moves on the infinite chain are matched by moves on the chain of size 2^n.

Case $\psi = [\chi]\tau$. Since the $\exists$-player has a winning strategy in $\mathcal{H}^{\psi}_{N_t}$, then she can choose a subset $X \subseteq S^{N'}$ such that $\ulcorner N'_{t'}, X, \chi, \tau \urcorner$ is a winning position. At the same time, she chooses $Y \subseteq S^{M'}$ in game $\mathcal{G}^{\psi}_{M_s}$, where $Y = \{s' \mid \exists t' \in X : B(s', t')\}$. By the IH, for all $s' \in Y$ and all $t' \in X$, we have $M'_{s'} \leftrightarrows^{2^n-k}_{P \setminus \{q\}} N'_{t'}$, which implies $M'_{s'} \leftrightarrows^{2^n-k-1}_{P \setminus \{q\}} N'_{t'}$. Observe that our construction of Y guarantees that for each state of X there is always a state of Y such that they are in relation B, and vice versa.

The $\forall$-player can now reply with one of three possible moves in both games. First, she can choose some state $s^* \in Y$ (resp. $t^* \in X$) to get to position $\ulcorner M'_{s^*}, \chi \urcorner$ (resp. $\ulcorner N'_{t^*}, \chi \urcorner$). That the bisimulation is preserved follows from the construction of Y and the IH. Similarly for the move of the universal player to position $\ulcorner M'_{s^*}, t(\neg\chi) \urcorner$ (resp. $\ulcorner N'_{t^*}, t(\neg\chi) \urcorner$). Finally, if the $\forall$-player chooses $\ulcorner M^Y_{s'}, \tau \urcorner$ (resp. $\ulcorner N^X_{t'}, \tau \urcorner$) in game $\mathcal{G}^{\psi}_{M_s}$ (resp. $\mathcal{H}^{\psi}_{N_t}$), then by the IH $M^Y_{s'} \leftrightarrows^{2^n-k}_{P \setminus \{q\}} N^X_{t'}$, which implies $M^Y_{s'} \leftrightarrows^{2^n-k-1}_{P \setminus \{q\}} N^X_{t'}$.

Cases $\psi = [!]\chi$ and $\psi = \langle ! \rangle \chi$. In game $\mathcal{G}^{\psi}_{M_s}$, the $\forall$-player makes a move, according to her winning strategy, from $\ulcorner M'_{s'}, [!]\chi \urcorner$ to some $\ulcorner M'_{s'}, [t(\tau)]\chi \urcorner$ such that $t(\tau) \in \mathcal{EL}$. It can be shown[2] [20, Theorem 8.15] that for each $d \in \mathbb{N}$, for all states $s' \in S^{M'}$ there is a state $t' \in S^{N'}$ (and vice versa) such that $M'_{s'} \models \varphi$ iff $N'_{t'} \models \varphi$ for all $\varphi \in \mathcal{EL}$ such that $md(\varphi) = d$. This fact in conjunction with $t(\tau) \in \mathcal{EL}$, entails that the universal player can choose the same formula in game $\mathcal{H}^{\psi}_{N_t}$ to move to a winning state $\ulcorner N'_{t'}, [t(\tau)]\chi \urcorner$. Note that the modal depth of $t(\tau)$ can exceed $2^n - k$. In this case, the games are continued with the current IH, and if the number of moves in a game exceeds $2^n - k$, then the game is continued with the assumption of $P \setminus \{q\}$-0-bisimilarity. It is enough for our purposes, since we are interested only in up to 2^n moves. Hence, we still have $M'_{s'} \leftrightarrows^{2^n-k}_{P \setminus \{q\}} N'_{t'}$ by the IH, which implies $M'_{s'} \leftrightarrows^{2^n-k-1}_{P \setminus \{q\}} N'_{t'}$.

The case of $\psi = \langle ! \rangle \chi$ is similar with the existential player as the protagonist.

Cases $\psi = [G]\chi$ and $\psi = \langle G \rangle \chi$ are similar to the cases above substituting $t(\tau)$ with $t(\tau_G)$, and $\mathcal{EL}$ with $\mathcal{EL}^G$.

As a result of these two simultaneous games over formula ψ and models M_s and N_t we end up in states in both games where the $\exists$-player (resp. the $\forall$-player) has a winning strategy. This contradicts the assumption that the $\forall$-player (resp. the $\exists$-player) has a winning strategy in one of the games, or, equivalently, it contradicts the fact that $M_s \not\models \psi$ iff $N_t \models \psi$. $\square$

Now we turn to the other direction of the expressivity relation. We use the same approach with formula games to show that, perhaps more surprisingly, there are some

[2] This is not the case for $\varphi \in \mathcal{ELC}$ for the same reasons as in the proof of Theorem 4.

properties of models that can be expressed by $\mathcal{APALC}$ and $\mathcal{GALC}$ and cannot be expressed by $\mathcal{APALC}^X$ and $\mathcal{GALC}^X$. Indeed, one may have expected that since quantifiers of $\mathcal{APALC}^X$ and $\mathcal{GALC}^X$ range over a strictly more expressive language than quantifiers of $\mathcal{APALC}$ and $\mathcal{GALC}$ ($\mathcal{ELC}$ in the first case, and $\mathcal{EL}$ in the second case), then $\mathcal{APALC}^X$ and $\mathcal{GALC}^X$ would end up being more expressive than their non-extended siblings. We show that this is not the case.

We start with providing two models and arguing that there are formulas of $\mathcal{APALC}$ and $\mathcal{GALC}$ that can distinguish them. Consider models M and N in Figure 6. In both of the models, there are vertical chains starting from s and t correspondingly of length $n+2$ for each $n \in \mathbb{N}$. These finite chains have at their end a numbered boxed state where q is true. Both models also have infinite vertical chains starting from u and v correspondingly. For the infinite chains, there are no states where q holds. Propositional variable p' is true only on the infinite chain of model N in the black square state.

Lemma 5. There are formulas $\psi_1 \in \mathcal{APALC}$ and $\psi_2 \in \mathcal{GALC}$, such that $M_s \not\models \psi_*$ and $N_t \models \psi_*$, where $* \in \{1, 2\}$.

Proof. Let φ be as in the proof of Theorem 7, and $\langle ! \rangle \varphi \in \mathcal{APALC}$. Similarly to the proof of Theorem 7, formula φ is satisfied in model O_t from Figure 5 and all models bisimilar to it. That $M_s \not\models \langle ! \rangle \varphi$ can be shown similarly to $M_s \not\models \langle ! \rangle^X \varphi$ (see proof of Theorem 7) noting that although upper and lower halves of M are not bisimialr, they nevertheless satisfy the same formulas of $\mathcal{EL}$ [20, Theorem 8.15].

We now argue that $N_t \models \langle ! \rangle \varphi$. Notice that p' holds on the infinite chain starting at state v. Since the quantification over announcements is implicit, we can use p' and q in announcements. Moreover, we can use announcement of arbitrary finite depth. Before giving the announcement that results in a model satisfying φ, we show how using p' we can specify a distinguishing formula for state v; such a formula will be true in v and nowhere else in the model. We can characterise states on the infinite chain by using their distance from p'. See Figure 7 for the representation of the approach.

Thus the distinguishing formula for v is

$$\chi_v := \Diamond_b \Diamond_i^m p' \land \Box_c \Box_j^m \neg p' \land \Diamond_a \Diamond_b p,$$

where $\Diamond_i^m$ stands for m alternating c- and b-diamonds, and $\Box_j^m$ stands for m alternating b- and c-boxes. Informally, the first conjunct means that state v is *at most* $m+1$ steps away from the p'-state, the second conjunct specifies that the state is *at least* $m+1$ steps away from the p'-state, and the third conjunct says that v is two steps away from the p-state.

Now we can use χ_v to provide the necessary formula. Consider the following announcement:

$$\psi_c := \Box_c((\neg p \to (\Box_{\{b,c\}} q \lor \Diamond_b p)) \land (q \to \Box_a \neg \chi_v)).$$

That the result of updating N_t with this formula is model O_t that satisfies φ can be shown similarly to the proof of Lemma 4 with state u being substituted by state v, and $\blacklozenge_{\{b,c\}} \neg q$ being substituted with $\neg \chi_v$. The argument for $\langle \{c\} \rangle \varphi \in \mathcal{GALC}$ also follows noting that $\psi_c \in \mathcal{EL}^{\{c\}}$. $\square$

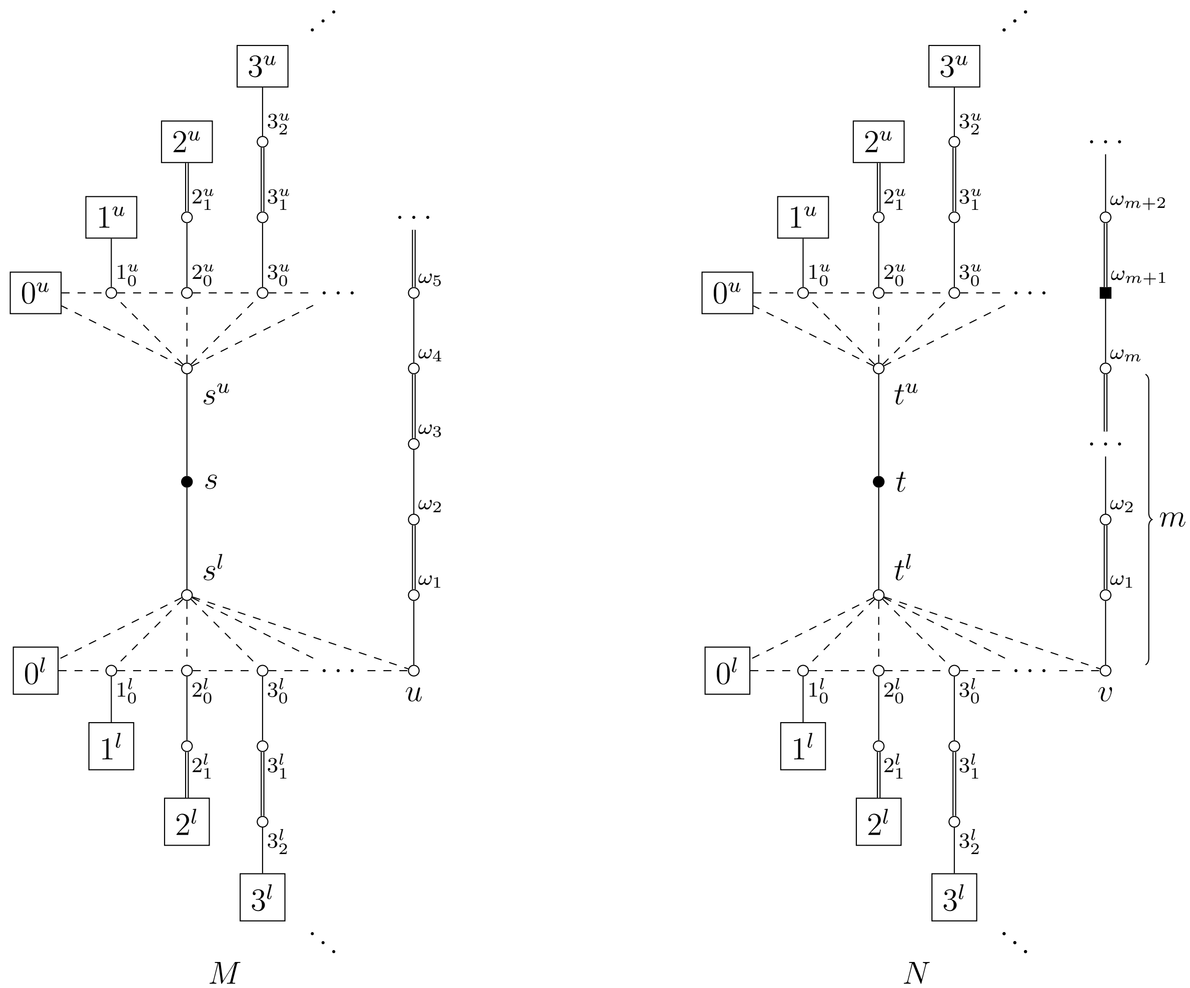

Figure 6: Models M and N. Relation for agent a is depicted by dashed lines, relation for b is shown by solid lines, and c's relations are double lines. Propositional variable p is true in black circle states, q is true in boxed states at the ends of chains, and p' is true in the black square state.

Before continuing with the expressivity proof, let us take another look at the two models. First, the reader can notice that they are $P \setminus \{p', q\}$- and $P \setminus \{p'\}$-bisimilar, and hence they satisfy the same formulas of $\mathcal{ELC}$ that do not contain p'. Second, all states on finite chains can be distinguished from all states on infinite chains. To see this, we show how to construct distinguishing formulas for each state on finite chains.

We can use the (slightly modified) method from Figure 7. First, we can construct a formula that is true only on a particular depth (number of steps from a p-state). For example, a formula that is true in all states that are exactly 4 steps away from a p-state is

$$\chi_4 := \Diamond_c \Diamond_b \Diamond_a \Diamond_b p \wedge \Box_{\{a,b\}} \Box_{\{a,c\}} \Box_{\{a,b\}} \Box_{\{a,b\}} \neg p.$$

The reader can verify that this formula holds in states, e.g., 3^l_2 and 3^u_2 of both models.

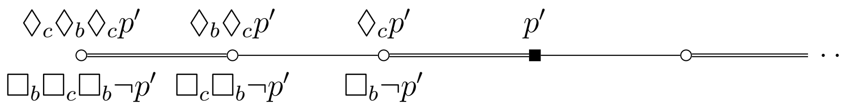

Figure 7: A segment of the infinite vertical chain from model N. A formula over or under a state means that the formula is true in the corresponding state.

To distinguish upper states from lower states, we, in addition to χ_4, need to use infinite chains. In model N_t or any submodel thereof containing the p'-state, we can use formula χ_v from the proof of Lemma 5. Thus, a formula that is true in all states that are exactly 4 steps away from a p-state and that are in the *lower* part of a models is

$$\chi_4^l := \Diamond_c\Diamond_b\Diamond_a\Diamond_b p \land \Box_{\{a,b\}}\Box_{\{a,c\}}\Box_{\{a,b\}}\Box_{\{a,b\}}\neg p \land \Diamond_c\Diamond_b\Diamond_a\chi_v.$$

The reader can check that $N_{3_2^l} \models \chi_4^l$ and $N_{3_2^u} \not\models \chi_4^l$. In order to choose states in the *upper* part of the model, we just negate the last conjunct. Thus,

$$\chi_4^u := \Diamond_c\Diamond_b\Diamond_a\Diamond_b p \land \Box_{\{a,b\}}\Box_{\{a,c\}}\Box_{\{a,b\}}\Box_{\{a,b\}}\neg p \land \neg\Diamond_c\Diamond_b\Diamond_a\chi_v.$$

Now we turn to distinguishing upper and lower parts of M_s and its submodels. Prima facie, it seems enough to use formula $\blacksquare_{\{b,c\}}\neg q$ that is true only in state u of M. However, if we want to deal also with submodels of M_s, it is not enough. Indeed, there may be some finite chains in some $M'_{s'}$ that do not have q-states at their ends, and that will thus satisfy $\blacksquare_{\{b,c\}}\neg q$. Hence, suppose that in some $M'_{s'}$ there is an infinite chain, and only a *finite* number of finite chains do not have q-states. Among those finite chains we take the longest, and denote its length by d. Now, a formula that is true only in state u is

$$\chi_u := \blacksquare_{\{b,c\}}\neg q \land \blacklozenge_{\{b,c\}}\neg\Diamond_i^{d+1}\Diamond_a\Diamond_b p,$$

where $\Diamond_i^{d+1}$ stands for $d+1$ alternating b- and c-diamonds. The first conjunct ensures that the formula is false on all chains with a q-state, and the second conjunct specifies that the formula is false on all chains with length less than $d+1$. Having defined χ_u, we can define a formula that would be true in all states that are exactly n steps away from a p-state and that are in the lower (or upper) part of the model. Formulas for states 4 steps away would be like χ_4^l and χ_4^u for N_t with χ_v being substituted with χ_u.

Finally, for the construction of the formula that is true only in 3_2^l, assume that χ_6^l and χ_5^l have been specified. Notice that 3_2^l is the only state in the lower parts of our models that is at depth 4, one step away from χ_5^l and does not reach a state satisfying χ_6^l on its chain. Formally,

$$\chi_{3_2^l} := \chi_4^l \land \Diamond_b\chi_5^l \land \Box_b\Box_c\neg\chi_6^l.$$

The described method of constructing distinguishing formulas of particular states will be used in the proof of Theorem 8.

Theorem 8. $\mathcal{APALC} \not\leq \mathcal{APALC}^X$ and $\mathcal{GALC} \not\leq \mathcal{GALC}^X$.

Proof. According to Lemma 5, there are formulas $\langle ! \rangle \varphi \in \mathcal{APALC}$ and $\langle \{c\} \rangle \varphi \in \mathcal{GALC}$ that distinguish models M_s and N_t. Now assume towards a contradiction that there is a $\psi \in \mathcal{APALC}^X \cup \mathcal{GALC}^X$ that is equivalent to either $\langle ! \rangle \varphi$ or $\langle \{c\} \rangle \varphi$ accordingly. Without loss of generality, we also assume that $\psi \in \mathcal{NNF}$. Since ψ has a finite number of symbols, there must be $q, p' \in P$ such that q and p' do not occur in ψ. Moreover, let $n = md(\psi)$.

Similarly to the proof of Theorem 7, we define a $P \setminus \{q, p'\}$-m-bisimulation relation $B \subseteq S_M \times S_N$:

$$B = \bigcup \left(\begin{array}{l} \{(s,t), (s^u, t^u), (s^l, s^l)\} \\ \{(m^*, m^*) \mid m \in \mathbb{N}, * \in \{u, l\}\} \\ \{(m^*_o, m^*_o) \mid m, o \in \mathbb{N}, o < m, * \in \{u, l\}\} \\ \{(u, v)\} \cup \{(\omega_k, \omega_k) \mid k \in \mathbb{N}\}\} \end{array} \right).$$

Relation B connects each state of M with the corresponding state of N.

Again, similarly to the proof of Theorem 7, we play two games simultaneously: game $\mathcal{G}^{\psi}_{M_s}$ over M_s and ψ, and game $\mathcal{H}^{\psi}_{N_t}$ over N_t and ψ. We also assume towards a contradiction that the $\forall$-player has a winning strategy in $\mathcal{G}^{\psi}_{M_s}$, and the $\exists$-player has a winning strategy in $\mathcal{H}^{\psi}_{N_t}$.

The proof for Boolean and epistemic cases, and the case of public announcements, follows the similar lines as the proof of Theorem 7, where the players play a move according to their winning strategy in one game, and play the corresponding move the other game. The crucial difference are the cases of quantified announcements.

Induction Hypothesis (IH): After k steps of a game, for all states s' and t' from all submodels M' and N', if $B(s', t')$, then $M'_{s'} \leftrightarroows^{2^n - k}_{P \setminus \{q, p'\}} N'_{t'}$.

Cases $\psi = [!]^X \chi$ and $\psi = \langle ! \rangle^X \chi$. In game $\mathcal{G}^{\psi}_{M_s}$, the $\forall$-player makes a move, according to her winning strategy, from $\ulcorner M'_{s'}, [!]\chi \urcorner$ to some $\ulcorner M'_{s'}, [t(\tau)]\chi \urcorner$ such that $t(\tau) \in \mathcal{ELC}$. Due to the construction of our models, we cannot guarantee that choosing $t(\tau)$ in $\mathcal{H}^{\psi}_{N_t}$ will result in $P \setminus \{q, p'\}$-$(2^n - k)$-bisimilar models, or, in other words, that it will also be a winning move for the universal player. However, as described earlier, we can construct a $\tau' \in \mathcal{ELC}$ such that $\ulcorner N'_{t'}, [t(\tau')]\chi \urcorner$ is a corresponding winning move in $\mathcal{H}^{\psi}_{N_t}$. Construction of τ' depends on the way the original τ updates M'. In particular, presence of the infinite chain in M' and of p'-state in N' allows us to distinguish upper and lower parts of the models. Thus, we need to take care that if one is affected, so is the other.

First, if in game $\mathcal{G}^{\psi}_{M_s}$ the $\forall$-player chooses such a τ that updating M' with the formula does not affect the infinite chain, does not remove an infinite number of q-states, and τ does not contain p', then she can make the same choice of τ in game $\mathcal{H}^{\psi}_{N_t}$ in position $\ulcorner N'_{t'}, [!]\chi \urcorner$. And vice versa for game $\mathcal{H}^{\psi}_{N_t}$. Such an announcement does not affect the ability to distinguish upper and lower halves of both models, thus retaining $P \setminus \{q, p'\}$-$(2^n - k - 1)$-bisimilarity.

Assume now that in game $\mathcal{G}^{\psi}_{M_s}$ formula τ contains p'. Since the valuation of p' in M is empty, we can get an equivalent τ' for game $\mathcal{H}^{\psi}_{N_t}$ by substituting p' in τ with $\bot$. This will ensure that updating M' with τ and updating N' with τ' results in $P \setminus \{q, p'\}$-$(2^n - k - 1)$-bisimilar models.

Let updating M' with τ remove an infinite number of q-states. As a result, we cannot distinguish states on the infinite chain from states on finite chains without q-states. In particular, for formulas χ_u with any d, there will a finite chain satisfying it. To model such an effect in N', the $\forall$-player chooses $\tau' := \tau \wedge \neg p'$, that removes the p'-state once being announced. As a result, we also lose the power to distinguish the upper and lower parts in N'. Moreover, since the p'-state is $2^n + 2$ away from t, we have $M'_{s'} \leftrightarrows^{2^n-k-1}_{P\setminus\{q,p'\}} N'_{t'}$.

Now consider game $\mathcal{H}^{\psi}_{N_t}$ and τ that, once being announced, removes an *infinite* number of q-states. Since the p'-state is still present in the updated model $(N')^{\tau}$, we need to retain the power to distinguish upper and lower parts in model M'. To this end, in game $\mathcal{G}^{\psi}_{M_s}$ the $\forall$-player chooses τ' announcement which would remove a *finite* number of q-states in M'. It is enough to consider only first 2^n chains. Since the number of states to remove is finite, the universal player can choose $\bigwedge \neg\chi_{i^l_j}$, where $\chi_{i^l_j}$ is a distinguishing formula of state i^l_j to be removed. This will preserve the power to distinguish upper and lower parts of the model using formulas χ_u for various d's, while also retaining $P \setminus \{q, p'\}$-$(2^n - k - 1)$-bisimilarity.

Finally, let updating M' with τ in game $\mathcal{G}^{\psi}_{M_s}$ cut the infinite chain to some finite length. In the resulting updated model, each state ω_i on now finite chain will be bisimilar to some state on a finite chain, thus making it impossible to distinguish upper and lower parts of the model. To simulate this in model N' in game $\mathcal{H}^{\psi}_{N_t}$, the $\forall$-player can choose $\tau' := \tau \wedge \neg p'$, thus making it impossible also in N' to distinguish upper and lower halves and maintaining the $P \setminus \{q, p'\}$-$(2^n - k - 1)$-bisimilarity.

If in game $\mathcal{H}^{\psi}_{N_t}$ the choice of τ is such that in the resulting update $(N')^{\tau}$ the infinite chain is cut, then we consider two cases. First, suppose that the chain was cut in such a way that ω_i is the last state of the now finite chain, and that the p'-state is still in $S^{(N')^{\tau}}$. Then we just need to cut a finite chain of length greater than 2^n (to maintain the $P \setminus \{q, p'\}$-$(2^n - k - 1)$-bisimilarity) in model M' to the same length i. This can be done by the $\forall$-player choosing $\bigwedge \neg\chi^l_j$, where χ^l_j are distinguishing formulas of states on the chosen finite chain. Second, if the chain was cut in such a way that ω_i is the last state of the now finite chain, and that the p'-state is *not* in $S^{(N')^{\tau}}$, then the infinite chain of M' should be cut to the same length. This can be done by the choice of $\Diamond^i_j \chi_u$ by the $\forall$-player, where $\Diamond^i_j$ is a stack of alternating b- and c-diamonds of the required size. In both models, the power to distinguish upper and lower parts will be gone, thus preserving the $P \setminus \{q, p'\}$-$(2^n - k - 1)$-bisimilarity.

The case of $\psi = \langle ! \rangle \chi$ can be shown by similar reasoning, substituting the $\forall$-player with the $\exists$-player.

Cases $\psi = [G]^X\chi$ and $\psi = \langle G \rangle^X \chi$. The method of constructing announcements described in the previous case can be also used for group announcements. The only difference is that chosen announcements are prefixed with $\Box_a$ for all $a \in G$. This is due to the fact that group announcements quantify over $\mathcal{ELC}^G$. If a group of agents cannot target a particular state, then they can announce a disjunction of formulas in their equivalence class. For example, agent b cannot announce a formula that will only be true 3^l_0: such a formula would be prefixed with $\Box_b$ and thus should also be satisfied in 3^l_1. Instead, agent b can announce $\Box_b(\chi_{3^l_0} \vee \chi_{3^l_1}) \in \mathcal{ELC}^{\{b\}}$ to target *both* 3^l_0 and 3^l_1.

As in the proof of Theorem 7, we play two simultaneous games over M_s and N_t that end up in states where the $\exists$-player (resp. the $\forall$-player) has a winning strategy. This contradicts the assumption that the $\forall$-player (resp. the $\exists$-player) has a winning strategy in the other model, or, equivalently, it contradicts the fact that $M_s \not\models \psi$ iff $N_t \models \psi$. $\square$

4.4 APALC and APALC^X relative to GALC and GALC^X

In this section we explore the relative expressivity of arbitrary and group announcements with common knowledge when pitched against one another. The results here are obtained by adapting the corresponding results on the relative expressivity of $\mathcal{APAL}$ and $\mathcal{GAL}$ [2, 25, 26]. Thus, we present only sketches and general intuitions of the proofs pointing an interested reader to the cited literature for additional details.

We start by claiming that the proof of Theorem 20 from [2] can be used to show that $\mathcal{GALC}$ and $\mathcal{GALC}^X$ are not at least as expressive as $\mathcal{APALC}$ and $\mathcal{APALC}^X$.

First, the authors of [2] consider an $\mathcal{APAL}$ formula $\langle ! \rangle(\Box_a p \land \neg \Box_b \Box_a p)$, and assume towards a contradiction that there is an equivalent formula φ of $\mathcal{GAL}$ not containing q. Then, models M_u and N_u from Figure 8 are considered, noting that $M_u \not\models \langle ! \rangle(\Box_a p \land \neg \Box_b \Box_a p)$ and $N_u \models \langle ! \rangle(\Box_a p \land \neg \Box_b \Box_a p)$. In particular, announcement of $p \lor \neg q$ makes $\Box_a p \land \neg \Box_b \Box_a p$ true in N_u (see Figure 8 and model $N^{p \lor \neg q}$). Since $p \lor \neg q \in \mathcal{ELC}$, we also have that $\langle ! \rangle^X(\Box_a p \land \neg \Box_b \Box_a p)$ is a distinguishing formula for M_u and N_u. Moreover, $\langle ! \rangle(\Box_a p \land \neg \Box_b \Box_a p) \in \mathcal{APALC}$ and $\langle ! \rangle^X(\Box_a p \land \neg \Box_b \Box_a p) \in \mathcal{APALC}^X$, and hence M_u and N_u are distinguishable by formulas of $\mathcal{APALC}$ and $\mathcal{APALC}^X$.

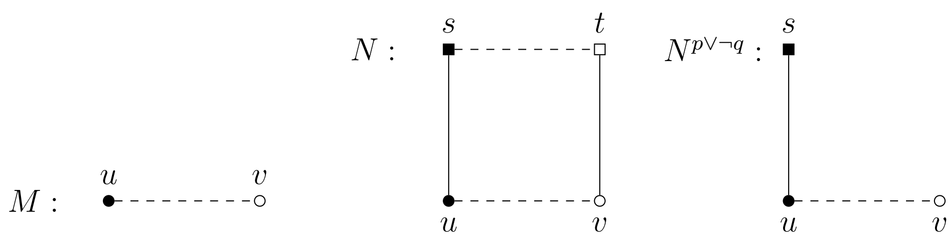

Figure 8: Models M, N, and $N^{p \lor \neg q}$. Relation for agent a is depicted by dashed lines and b's relation is shown by solid lines. Propositional variable p is true in black states, and propositional variable q is true in square states.

The argument that φ cannot distinguish M_u and N_u goes by induction [2, Theorem 20]. For our goals, it is enough to notice that M_u and N_u are $P \setminus \{q\}$-bisimilar and thus satisfy the same formulas of $\mathcal{PALC}$ that do not contain q. Moreover, cases for extended arbitrary and group announcements follow from the fact that M and N are finite, and thus by Theorem 2 satisfy $[G]\chi$ if and only if they satisfy $[G]^X\chi$.

Theorem 9. $\mathcal{APALC} \not\leq \mathcal{GALC}$, $\mathcal{APALC} \not\leq \mathcal{GALC}^X$, $\mathcal{APALC}^X \not\leq \mathcal{GALC}$, and $\mathcal{APALC} \not\leq \mathcal{GALC}^X$.

The fact that $\mathcal{GALC}$ and $\mathcal{GALC}^X$ are not at least as expressive as $\mathcal{APALC}$ and $\mathcal{APALC}^X$ follows from the proof of $\mathcal{GAL} \not\leq \mathcal{CAL}$ [25, 26], where $\mathcal{CAL}$ is the language of coalition announcement logic defined by

$$\mathcal{CAL} \ni \varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box_a \varphi \mid [\varphi]\varphi \mid [\langle G \rangle]\varphi.$$

The semantics of coalition announcement modality $[\langle G \rangle]\varphi$ and its dual $\langle [G] \rangle\varphi$ is as follows:

$$M_s \models [\langle G \rangle]\varphi \quad \text{iff} \quad M_s \models \psi_G \to \langle \psi_G \wedge \chi_{A \setminus G} \rangle \varphi \text{ for all } \psi_G \in \mathcal{EL}^G \text{ and some } \chi_{A\setminus G} \in \mathcal{EL}^{A \setminus G}$$
$$M_s \models \langle [G] \rangle\varphi \quad \text{iff} \quad M_s \models \psi_G \wedge [\psi_G \wedge \chi_{A \setminus G}] \varphi \text{ for some } \psi_G \in \mathcal{EL}^G \text{ and all } \chi_{A\setminus G} \in \mathcal{EL}^{A \setminus G}.$$

Informally, formula $\langle [G] \rangle\varphi$ means that agents from G have a joint announcements such that no matter what agents from outside of G announce at the same time, φ will hold. Similarly, $[\langle G \rangle]\varphi$ stands for the fact that whatever agents from G jointly announce, there is a counter-announcement by agents from outside of G such that φ will hold.

For the purposes at hand, we are interested in a special case of coalition announcements, namely announcement by the grand coalition A. In such a case, the semantics can be simplified to

$$M_s \models [\langle A \rangle]\varphi \quad \text{iff} \quad M_s \models [\psi_A]\varphi \text{ for all } \psi_A \in \mathcal{EL}^G$$
$$M_s \models \langle [A] \rangle\varphi \quad \text{iff} \quad M_s \models \langle \psi_A \rangle\varphi \text{ for some } \psi_A \in \mathcal{EL}^G.$$

The proof in [25, 26] starts off by presenting two classes of *finite* models, called A-chain models and B-chain models. Examples of chain models are depicted in Figure 9. Without giving a formal definition, we just mention that chain models have a leftmost state that satisfies $\neg p \wedge \Box_a \neg p$, and the rightmost state that satisfies $\Box_a p \wedge [A](\Diamond_b \neg p \to \Box_a \Diamond_b \neg p)$. In short, the models are similar in their extremities and differ only in length (see Figure 9 for reference).

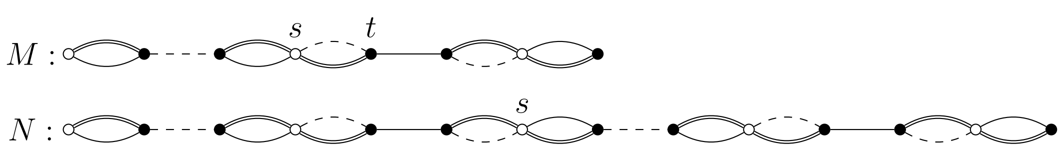

Figure 9: Chain models M and N. Relation for agent a is depicted by dashed lines and b's relation is shown by solid lines, and c's relations are double lines. Propositional variable p is true in black states.

Whether a given pointed chain model is an A-chain model or a B-chain model depends which agent relation is the first one in the direction of the state satisfying $\neg p \wedge \Box_a \neg p$: a-relation or b-relation. For example, model M_s from Figure 9 is a B-model since b's relation is the first one among a and b in the direction of the $\neg p \wedge \Box_a \neg p$-state (leftmost state). On the other hand, M_t and N_s are A-models.

Next, it is shown in [25, 26] that there is a formula of GAL φ such that for all M_s, $M_s \models \varphi$ if and only if M_s is an A-model. Hence, the same formula also belongs to the language of GALC. We do not present the formula since it is a bit involved and not essential for our argument here. To get a corresponding distinguishing formula of GALC^X, we first note that φ contains the following group announcement operators: $\langle c \rangle \chi$, $[c]\chi$, and $[\{a, b, c\}]\chi$. Since chain models are finite, by Theorem 2 we can equivalently substitute all occurrences of the abovementioned group announcements with $\langle c \rangle^X \chi$, $[c]^X \chi$, and $[\{a, b, c\}]^X \chi$ respectively.

After that, the authors of [25, 26] use formula games for $\mathcal{GAL}$ and $\mathcal{CAL}$ to show that no formula of $\mathcal{CAL}$ can distinguish the classes of A- and B-chain models. The proof follows a similar approach as we used in proofs of Theorems 7 and 8 in this paper. In particular, it is assumed towards a contradiction that for all A-chain models M_s there is a $\varphi \in \mathcal{CAL}$ such that $M_s \models \varphi$, and for all B-chain models N_t it holds that $N_t \not\models \varphi$. The proof proceeds by playing simultaneous formula games over $2^{md(\varphi)}$-bisimilar pointed A- and B-models, and the invariant that after i game steps, models are still $2^{md(\varphi)} - i$-bisimilar, is maintained.

We can reuse the proof to show that formulas of $\mathcal{APALC}$ and $\mathcal{APALC}^X$ do not distinguish A- and B-chain models. For the cases of $\blacksquare_G \chi$ and $\blacklozenge_G \chi$, the current player chooses a G-reachable state in one model, and the corresponding state in the other model. By 'corresponding' we mean a state which lies on the same distance from the closest extremity, i.e. from the leftmost or the rightmost state depending on which one is closer. In such a way we ensure that games continue in $2^{md(\varphi)} - i$-bisimilar models.

Cases $[!]\chi$ and $\langle ! \rangle \chi$ follow from cases $\langle\!\lbrack \{a, b, c\} \rbrack\!\rangle \chi$ and $\lbrack\!\langle \{a, b, c\} \rangle\!\rbrack \chi$ from [25, 26] noting that the intersection of a-, b-, and c-relations in chain models is an identity relation. This implies that the grand coalition $\{a, b, c\}$ can force *any* submodel of a given chain model with their announcements. Thus, coalition announcement for chain models $\langle\!\lbrack \{a, b, c\} \rbrack\!\rangle \chi$ is equivalent to $\langle ! \rangle \chi$ and $\lbrack\!\langle \{a, b, c\} \rangle\!\rbrack \chi$ is equivalent to $[!]\chi$. Since chain models are finite, we also have the result for $\langle ! \rangle^X \chi$ and $[!]^X \chi$ by Theorem 2.

Theorem 10. *$\mathcal{GALC} \not\leq \mathcal{APALC}$, $\mathcal{GALC} \not\leq \mathcal{APALC}^X$, $\mathcal{GALC}^X \not\leq \mathcal{APALC}$ and $\mathcal{GALC}^X \not\leq \mathcal{APALC}^X$.*

5 Proof system

In this section we start with the presentation of a proof system of GALC and a detailed completeness proof for it. We then discuss how both are modified to get corresponding results for GALC^X, APALC, and APALC^X.

Let us first introduce an auxiliary notion.

Definition 5.1. Let $\varphi \in \mathcal{GALC}$, $a \in A$, $G \subseteq A$, and $\sharp \notin P$. The set of *necessity forms* [30] is defined recursively below:

$$\eta(\sharp) ::= \sharp \mid \varphi \to \eta(\sharp) \mid \Box_a \eta(\sharp) \mid [\varphi]\eta(\sharp)$$

We will denote the result of replacing of $\sharp$ with φ in a necessity form $\eta(\sharp)$ as $\eta(\varphi)$.

Definition 5.2. The proof system of GALC is the following extension of the proof system of GAL [2]:

$(A0)$	Theorems of propositional logic
$(A1)$	$\Box_a(\varphi \to \psi) \to (\Box_a \varphi \to \Box_a \psi)$
$(A2)$	$\Box_a \varphi \to \varphi$
$(A3)$	$\Box_a \varphi \to \Box_a \Box_a \varphi$
$(A4)$	$\neg \Box_a \varphi \to \Box_a \neg \Box_a \varphi$
$(A5)$	$[\psi]p \leftrightarrow (\psi \to p)$
$(A6)$	$[\psi]\neg\varphi \leftrightarrow (\psi \to \neg[\psi]\varphi)$
$(A7)$	$[\psi](\varphi \wedge \chi) \leftrightarrow ([\psi]\varphi \wedge [\psi]\chi)$
$(A8)$	$[\psi]\Box_a \varphi \leftrightarrow (\psi \to \Box_a[\psi]\varphi)$
$(A9)$	$\blacksquare_G \varphi \to \Box_G^n \varphi$ for any $n \in \mathbb{N}$
$(A10)$	$[G]\varphi \to [\psi_G]\varphi$ for any $\psi_G \in \mathcal{EL}^G$
MP	From $\varphi \to \psi$ and φ, infer ψ
NK	From φ, infer $\Box_a \varphi$
NA	From φ, infer $[\psi]\varphi$
IC	From $\{\eta(\Box_G^n \varphi) \mid n \in \mathbb{N}\}$, infer $\eta(\blacksquare_G \varphi)$
IG	From $\{\eta([\psi_G]\varphi) \mid \psi_G \in \mathcal{EL}^G\}$, infer $\eta([G]\varphi)$.

We call GALC the minimal set that contains axioms $A0$–$A10$ and is closed under MP, NK, NA, IC, and IG.

Like existing complete systems of APAL and GAL [9, 21], this proof system of GALC is infinitary as it has inference rules that require an infinite number of premises. Note that one of them is the infinitary rule for common knowledge, which is less standard than the usual fixed point approach (see, for example, [11], and also [33] for an alternative axiomatisation of ELC). In an already infinitary system, this treatment is both more intuitive and technically simpler. The infinitary approach to common knowledge has also been discussed in [5], where the authors consider a corresponding Gentzen-type system.

Lemma 6. *IC* and *IG* are truth preserving.

Proof. The proof is a straightforward induction on necessity forms with the application of the definition of semantics. □

Necessitation rules for common knowledge and group announcements are derivable in GALC.

Lemma 7. Rules 'From φ, infer $\blacksquare_G \varphi$' and 'From φ, infer $[G]\varphi$' are derivable in GALC.

Proof. Given φ, we can use NK to derive $\Box_G^n \varphi$ for each n. Since formulas $\Box_G^n \varphi$ are in the necessity form, we can apply IC to infer $\blacksquare_G \varphi$. Similarly, we can infer $[\psi_G]\varphi$ for each $\psi_G \in \mathcal{EL}^G$ using rule NA. After that, application of IG results in $[G]\varphi$. □

Theorem 11. GALC is sound.

Proof. Due to the soundness of GAL, Lemma 6, and the validity of ($A9$). $\square$

In order to prove the completeness, we adapt the completeness proof of APAL from [9, 10, 8].

Whenever we will use induction on the formula structure of some $\varphi \in \mathcal{GALC}$, we will use the following measure.

Definition 5.3. Let $\varphi \in \mathcal{GALC}$. The $\blacksquare$-*depth* $\delta_\blacksquare(\varphi)$ of φ is defined similarly to the quantifier depth $\delta_\forall$ (Definition 2.3) with the following exceptions:

$$\delta_\blacksquare([G]\varphi) = \delta_\blacksquare(\varphi) \qquad\qquad \delta_\blacksquare(\blacksquare_G\varphi) = \delta_\blacksquare(\varphi) + 1$$

The *complexity* $c(\varphi)$ of φ is

$$\begin{aligned} c(p) &= 1 & c([\psi]\varphi) &= c(\psi) + 3 \cdot c(\varphi) \\ c(\neg\varphi) = c(\Box_a\varphi) &= c(\varphi) + 1 & c(\blacksquare_G\varphi) &= c(\varphi) + 1 \\ c(\varphi \land \psi) &= \max(c(\varphi), c(\psi)) + 1 & c([G]\varphi) &= c(\varphi) + 1 \end{aligned}$$

Let $\varphi, \psi \in \mathcal{GALC}$. We have that $\varphi <^\forall_\blacksquare \psi$ if and only if $\delta_\forall(\varphi) < \delta_\forall(\psi)$, or, otherwise, $\delta_\forall(\varphi) = \delta_\forall(\psi)$, and either $\delta_\blacksquare(\varphi) < \delta_\blacksquare(\psi)$, or $\delta_\blacksquare(\varphi) = \delta_\blacksquare(\psi)$ and $c(\varphi) < c(\psi)$. Relation $<^\forall_\blacksquare$ is a well-founded partial order.

Lemma 8. Let $\varphi, \psi, \chi \in \mathcal{GALC}$ and $G \subseteq A$. The following inequalities hold:

$$\begin{aligned} &\varphi <^\forall_\blacksquare \neg\varphi & [\psi]\varphi \land [\psi]\chi &<^\forall_\blacksquare [\psi](\varphi \land \chi) \\ &\varphi <^\forall_\blacksquare \varphi \land \psi & [\psi]\Box^n_G\varphi &<^\forall_\blacksquare [\psi]\blacksquare_G\varphi \\ &\varphi <^\forall_\blacksquare \Box_a\varphi & [\psi][\psi_G]\varphi &<^\forall_\blacksquare [\psi][G]\varphi \\ &p <^\forall_\blacksquare [\psi]p & \Box^n_G\varphi &<^\forall_\blacksquare \blacksquare_G\varphi \\ &\psi \to \neg[\psi]\varphi <^\forall_\blacksquare [\psi]\neg\varphi & [\psi_G]\varphi &<^\forall_\blacksquare [G]\varphi \end{aligned}$$

Our completeness proof is based on the canonical model construction. We will use *theories* as states in the canonical model.

Definition 5.4. A set x is called a *theory* if it contains all theorems and is closed under MP, IC, and IG. The smallest theory is GALC. Theory x is *consistent* if there is no $\varphi \in \mathcal{GALC}$ such that $\varphi, \neg\varphi \in x$. Theory x is *maximal* if any theory y such that $x \subset y$ is inconsistent.

Lemma 9. Theory x is maximal if and only if for all $\varphi \in \mathcal{GALC}$ we have that either $\varphi \in x$ or $\neg\varphi \in x$.

Proof. Let x be a maximal theory, and assume towards a contradiction that there is a $\varphi \in \mathcal{GALC}$ such that neither $\varphi \in x$ nor $\neg\varphi \in x$. Then theory $x \cup \{\varphi\}$ is consistent, and $x \subset x \cup \{\varphi\}$, which contradicts the definition of maximality.

In the other direction, let us for all $\varphi \in \mathcal{GALC}$ have that either $\varphi \in x$ or $\neg\varphi \in x$, and x be not maximal. This implies that there is a consistent y such that $x \subset y$, and in particular that there is a ψ such that $\psi \notin x$ and $\psi \in y$. Since y is consistent, $\neg\psi \notin y$, and hence $\neg\psi \notin x$. We now have that $\psi \notin x$ and $\neg\psi \notin x$ that contradicts our assumption. $\square$

Lemma 10. Let $\varphi, \psi \in \mathcal{GALC}$, if x is a theory, then $x + \varphi := \{\chi \mid \varphi \to \chi \in x\}$, $\Box_a x := \{\chi \mid \Box_a \chi \in x\}$, and $[\psi]x := \{\chi \mid [\psi]\chi \in x\}$ are theories as well. Also, $x + \varphi$ is consistent if and only if $\neg\varphi \notin x$.

Proof. An extension of the proof of Lemma 4.11 in [9], where common knowledge cases are dealt with using $(A9)$ and IC. $\square$

Lemma 11. For all theories x and all $\varphi \in \mathcal{GALC}$, it holds that $x \subseteq x + \varphi$.

Proof. Let us for some $\psi \in \mathcal{GALC}$ have that $\psi \in x$. Since x is a theory and thus contains all the instances of propositional tautologies, $\psi \to (\varphi \to \psi) \in x$. As x is closed under MP, $\varphi \to \psi \in x$, and, by Lemma 10, $\psi \in x + \varphi$. $\square$

Next, we prove the Lindenbaum lemma.

Lemma 12. If x is a consistent theory, then it can be extended to a maximal consistent theory y such that $x \subseteq y$.

Proof. The proof is a variation of the Lindenbaum Lemma for APAL [9, Lemma 4.12]. We give here a sketch of an extended proof.

Let $\{\varphi_0, \varphi_1, \ldots\}$ be an enumeration of formulas of $\mathcal{GALC}$, and let $y_0 = x$. Assume that for some $n \geq 0$, $x \subseteq y_n$ and y_n is a consistent theory. If $\neg\varphi_n \notin y_n$, then $y_{n+1} = y_n + \varphi_n$. Otherwise, there are three cases to consider.

First, if $\neg\varphi_n \in y_n$ and φ_n is not of either the form $\eta(\blacksquare_G \psi)$ or the form $\eta([G]\psi)$, then $y_{n+1} = y_n$. Second, if $\neg\varphi_n \in y_n$ and φ_n is of the form $\eta(\blacksquare_G \psi)$, then $y_{n+1} = y_n + \neg\eta(\Box_G^n \psi)$, where $\neg\eta(\Box_G^n \psi)$ is the first formula in the enumeration such that $\eta(\Box_G^n \psi) \notin y_n$. Third, if $\neg\varphi_n \in y_n$ and φ_n is of the form $\eta([G]\psi)$, then $y_{n+1} = y_n + \neg\eta([\psi_G]\psi)$, where $\neg\eta([\psi_G]\psi)$ is the first formula in the enumeration such that $\eta([\psi_G]\psi) \notin y_n$.

In all these cases it is clear that y_{n+1} is consistent. Also, using the inductive construction of y_{n+1}, the fact that $x \subseteq y_{n+1}$, it is relatively straightforward to show that $y = \bigcup_{n=0}^{\infty} y_n$ is a maximal consistent theory such that $x \subseteq y$. $\square$

Now we are ready to define the canonical model, where states are maximal consistent theories.

Definition 5.5. We call model $\mathfrak{M} = (\mathfrak{S}, \mathfrak{R}, \mathfrak{V})$, where $\mathfrak{S} = \{x \mid x \text{ is a maximal consistent theory}\}$, $\mathfrak{R}(a) = \{(x, y) \mid \Box_a x \subseteq y\}$, and $\mathfrak{V}(p) = \{x \mid p \in x\}$, the *canonical model.*

Next, we prove the truth lemma.

Lemma 13. For all maximal consistent theories x and $\varphi \in \mathcal{GALC}$, $\varphi \in x$ if and only if $\mathfrak{M}_x \models \varphi$.

Proof. Proofs for boolean, epistemic, some of public announcement cases are quite similar to those in [10, Lemma 11], and can be shown using the axioms of GALC and Lemma 8. We show here only the cases that include group announcements and common knowledge.

Induction hypothesis (IH): For all maximal consistent theories y and formulas $\psi \in \mathcal{GALC}$, if $\psi <^{\forall}_{\blacksquare} \varphi$, then $\psi \in y$ iff $\mathfrak{M}_y \models \psi$.

Case $\varphi = [\chi]\blacksquare_G\psi$. ($\Rightarrow$): Suppose that $[\chi]\blacksquare_G\psi \in x$. Since x contains all theorems of GALC, we have for all $n \in \mathbb{N}$, $[\chi](\blacksquare_G\psi \to \Box^n_G\psi) \in x$ and $[\chi](\blacksquare_G\psi \to \Box^n_G\psi) \to ([\chi]\blacksquare_G\psi \to [\chi]\Box^n_G\psi) \in x$ (Proposition 4.46.3 of [20]). Using *MP* twice, we get $[\chi]\Box^n_G\psi \in x$ for all $n \in \mathbb{N}$. By the IH, this is equivalent to $\forall n \in \mathbb{N} : \mathfrak{M}_x \models [\chi]\Box^n_G\psi$. The latter is equivalent to the fact that $\mathfrak{M}_x \models \chi$ implies $\mathfrak{M}^\chi_x \models \Box^n_G\psi$ for all n. By the semantics of common knowledge we have that $\mathfrak{M}_x \models \chi$ implies $\mathfrak{M}^\chi_x \models \blacksquare_G\psi$, and the latter is $\mathfrak{M}_x \models [\chi]\blacksquare_G\psi$ by the semantics of public announcements.

($\Leftarrow$): Assume that $\mathfrak{M}_x \models [\chi]\blacksquare_G\psi$. By the semantics, this is equivalent to the fact that $\mathfrak{M}_x \models \chi$ implies $\mathfrak{M}^\chi_x \models \blacksquare_G\psi$. By the semantics of common knowledge, the latter is $\forall n \in \mathbb{N} : \mathfrak{M}^\chi_x \models \Box^n_G\psi$. We can 'fold' the public announcement back: $\forall n \in \mathbb{N} : \mathfrak{M}_x \models [\chi]\Box^n_G\psi$. By the IH, $\forall n \in \mathbb{N} : [\chi]\Box^n_G\psi \in x$. Observe that this formula is in a necessity form. Hence, we conclude, by rule *IC*, that $[\chi]\blacksquare_G\psi \in x$.

Case $\varphi = [\chi][G]\tau$. ($\Rightarrow$): Suppose that $[\chi][G]\tau \in x$. Since x contains all theorems of GALC, we have for all $\psi_G \in \mathcal{EL}^G$, $[\chi]([G]\tau \to [\psi_G]\tau) \in x$ and $[\chi]([G]\tau \to [\psi_G]\tau) \to ([\chi][G]\tau \to [\chi][\psi_G]\tau) \in x$ (Proposition 4.46.3 of [20]). Using *MP* twice, we get $[\chi][\psi_G]\tau \in x$ for all $\psi_G \in \mathcal{EL}^G$. By the IH, this is equivalent to $\forall \psi_G \in \mathcal{EL}^G : \mathfrak{M}_x \models [\chi][\psi_G]\tau$. The latter is equivalent to the fact that $\mathfrak{M}_x \models \chi$ implies $\mathfrak{M}^\chi_x \models [\psi_G]\tau$ for all $\psi_G \in \mathcal{EL}^G$. By the semantics of group announcements we have that $\mathfrak{M}_x \models \chi$ implies $\mathfrak{M}^\chi_x \models [G]\tau$, and the latter is $\mathfrak{M}_x \models [\chi][G]\tau$ by the semantics of public announcements.

($\Leftarrow$): Assume that $\mathfrak{M}_x \models [\chi][G]\tau$. By the semantics, this is equivalent to the fact that $\mathfrak{M}_x \models \chi$ implies $\mathfrak{M}^\chi_x \models [G]\tau$. By the semantics of group announcements, the latter is $\forall \psi_G \in \mathcal{EL}^G : \mathfrak{M}^\chi_x \models [\psi_G]\tau$. We can 'fold' the public announcement back: $\forall \psi_G \in \mathcal{EL}^G : \mathfrak{M}_x \models [\chi][\psi_G]\tau$. By the IH, $\forall \psi_G \in \mathcal{EL}^G : [\chi][\psi_G]\tau \in x$. Observe that this formula is in a necessity form. Hence, we conclude, by rule *IG*, that $[\chi][G]\tau \in x$.

Case $\varphi = \blacksquare_G\psi$. ($\Rightarrow$): Assume that $\blacksquare_G\psi \in x$. By (*A9*), $\forall n \in \mathbb{N} : \Box^n_G\psi \in x$, which is equivalent, by the IH, to $\forall n \in \mathbb{N} : \mathfrak{M}_x \models \Box^n_G\psi$. This is equivalent to $\mathfrak{M}_x \models \blacksquare_G\psi$ by the semantics.

($\Leftarrow$): Assume that $\mathfrak{M}_x \models \blacksquare_G\varphi$. By the semantics, this is equivalent to $\forall n \in \mathbb{N} : \mathfrak{M}_x \models \Box^n_G\varphi$. Furthermore, by the IH, we have $\forall n \in \mathbb{N} : \Box^n_G\varphi \in x$. Since x is closed under *IC*, we have $\blacksquare_G\varphi \in x$.

Case $\varphi = [G]\chi$. ($\Rightarrow$): Assume that $[G]\chi \in x$. By (*A10*), $\forall \psi_G \in \mathcal{EL}^G : [\psi_G]\chi \in x$, which is equivalent, by the IH, to $\forall \psi_G \in \mathcal{EL}^G : \mathfrak{M}_x \models [\psi_G]\chi$. This is equivalent to $\mathfrak{M}_x \models [G]\chi$ by the semantics.

($\Leftarrow$): Assume that $\mathfrak{M}_x \models [G]\chi$. By the semantics, this is equivalent to $\forall \psi_G \in \mathcal{EL}^G : \mathfrak{M}_x \models [\psi_G]\varphi$. Furthermore, by the IH, we have $\forall \psi_G \in \mathcal{EL}^G : [\psi_G]\varphi \in x$. Since x is closed under IG, we can infer that $[G]\chi \in x$. $\square$

Finally, we can prove the completeness of GALC.

Theorem 12. For all $\varphi \in \mathcal{GALC}$, if φ is valid, then $\varphi \in$ GALC.

Proof. Assume towards a contradiction that φ is valid and $\varphi \notin$ GALC. Since GALC is a consistent theory, it follows that GALC $+ \neg\varphi$ is a consistent theory as well. By Lemma 5, there is a maximal consistent theory x such that GALC $+ \neg\varphi \subseteq x$. By Lemma 11, $\neg\varphi \in$ GALC $+ \neg\varphi$, and hence $\neg\varphi \in x$. Since x is a maximal consistent theory, it follows that $\varphi \notin x$. According to Lemma 13, $\varphi \notin x$ is equivalent to $\mathfrak{M}_x \not\models \varphi$, which contradicts φ being valid. $\square$

The proof system of GALC^X is the same as in Definition 5.2 with following differences:

$$\begin{array}{rl} (A10)' & [G]^X\varphi \to [\psi_G]\varphi \text{ for any } \psi_G \in \mathcal{ELC}^G \\ IG' & \text{From } \{\eta([\psi_G]\varphi) \mid \psi_G \in \mathcal{ELC}^G\}, \text{ infer } \eta([G]^X\varphi). \end{array}$$

The completeness proof is exactly as for GALC, with each $[G]$ replaced by $[G]^X$ and $\mathcal{EL}^G$ replaced by $\mathcal{ELC}^G$.

Theorem 13. GALC^X is sound and complete.

The axiomatisation of APALC is the same as the proof system of GALC with the following differences:

$$\begin{array}{rl} (A10)' & [!]\varphi \to [\psi]\varphi \text{ for any } \psi_G \in \mathcal{EL} \\ IG' & \text{From } \{\eta([\psi]\varphi) \mid \psi \in \mathcal{EL}\}, \text{ infer } \eta([!]\varphi). \end{array}$$

Again, the completeness proof is exactly the same as for GALC, replacing $[G]$ with $[!]$ and each $\mathcal{EL}^G$ with $\mathcal{EL}$.

Theorem 14. APALC is sound and complete.

Finally, the proof system and the completeness of APALC^X can be obtained from those of APALC in the same way as for GALC^X.

Theorem 15. APALC^X is sound and complete.

6 Common Knowledge in Quantification Over Information Change

The way we dealt with common knowledge in Section 5 to get the completeness results is quite idiosyncratic. As the reader may have already observed, we treated common knowledge as an infinitary modality. Moreover, the proof systems we provided did not require any specific interaction axioms for common knowledge. Thus the proof can be used to establish completeness of extensions of various other logics of quantified information change.

6.1 Boolean and Positive Announcements

The undecidability result for APAL [4] spurred the quest for finding decidable fragments of the logic. In particular, the question was whether more modest versions of quantification lead to decidability. It was answered positively for (at least) two versions of APAL: the one, where [!] ranges over Boolean formulas, and the one, where [!] ranges over positive PAL formulas.

The language and the semantics of *Boolean APAL* (BAPAL) [16] are quite similar to those of APAL with the only difference in the interpretation of $[!]\varphi$:

$$M_s \models [!]_{\mathrm{BAPAL}}\varphi \quad \text{iff} \quad M_s \models [\psi]\varphi \text{ for all } \psi \in \mathcal{PL},$$

where $\mathcal{PL}$ is the language of propositional logic.

BAPAL is quite a unique logic in the family of the logics of quantified announcements, since it has a finitary axiomatisation, it is decidable [18], and yet lacks the finite model property. Contrast this to the undecidability [4] and the lack of finite model property [17] for the standard logics of quantified announcements.

Alongside the finitary axiomatisation of BAPAL, the authors of [16] also provide an infinitary one, and using the latter we can give an axiomatisation of *BAPAL with common knowledge* (BAPALC).

The proof system of (the infinitary version of) BAPALC is the same as in Definition 5.2 with the following differences:

$$\begin{array}{rl} (A10)' & [!]_{\mathrm{BAPAL}}\varphi \to [\psi]\varphi \text{ for any } \psi \in \mathcal{PL} \\ IG' & \text{From } \{\eta([\psi]\varphi) \mid \psi \in \mathcal{PL}\}, \text{ infer } \eta([!]_{\mathrm{BAPAL}}\varphi). \end{array}$$

Theorem 16. BAPALC is sound and complete.

The completeness proof follows the one in Section 5 with $[G]$ being substituted with $[!]_{\mathrm{BAPAL}}$, and $\mathcal{EL}^G$ being replaced by $\mathcal{PL}$.

Positive APAL (PAPAL) [19], similarly to BAPAL, restricts the range of quantification of arbitrary public announcement operators:

$$M_s \models [!]_{\mathrm{PAPAL}}\varphi \quad \text{iff} \quad M_s \models [\psi]\varphi \text{ for all } \psi \in \mathcal{EL}^+.$$

As with APAL and GAL, extending PAPAL with common knowledge can be done in (at least) two meaningful ways: we can add common knowledge to the language but leave the semantics of $[!]_{\mathrm{PAPAL}}\varphi$ intact, or we can also extend the quantification to a larger fragment with common knowledge. The resulting logic is *PAPAL with common knowledge*, and we will denote the former variant as PAPALC and the latter variant as PAPALC^X. The semantics of the quantifier in PAPALC^X is the following:

$$M_s \models [!]^X_{\mathrm{PAPAL}}\varphi \quad \text{iff} \quad M_s \models [\psi]\varphi \text{ for all } \psi \in \mathcal{ELC}^+.$$

The axiomatisation of PAPALC is yet again a variation of the proof system for GALC with substitutions:

$$(A10)' \quad [!]_{\mathrm{PAPAL}}\varphi \to [\psi]\varphi \text{ for any } \psi \in \mathcal{EL}^+$$

$$IG' \quad \text{From } \{\eta([\psi]\varphi) \mid \psi \in \mathcal{EL}^+\}, \text{ infer } \eta([!]_{\text{PAPAL}}\varphi).$$

To get the axiomatisation of PAPALC^X it is enough to change $\mathcal{EL}^+$ to $\mathcal{ELC}^+$, and $[!]_{\text{PAPAL}}$ to $[!]^X_{\text{PAPAL}}$ in $(A10)'$ and IG'.

Theorem 17. Both PAPALC and PAPALC^X are sound and complete.

For both systems completeness can be shown in a similar fashion to the completeness of GALC with $[G]$ being substituted with $[!]_{\text{PAPAL}}$ (with $[!]^X_{\text{PAPAL}}$ for PAPALC^X), and with each $\mathcal{EL}^G$ replaced by $\mathcal{EL}^+$ (by $\mathcal{ELC}^+$ for PAPALC^X).

6.2 Coalition announcements

The results for GALC and APALC go hand-in-hand with each other due to the fact that the underlying logics are relatively similar. So far we have omitted from the discussion, however, an interesting cousin of GAL and APAL, *coalition announcement logic* (CAL) [3, 26]. CAL extends PAL with the modality $[\langle G\rangle]\varphi$, meaning 'whatever agents from group G announce, there is a simultaneous counter-announcement by the agents from outside of the group such that φ holds in the resulting model'. It is clear that modalities $[\langle G\rangle]\varphi$ are game-theoretical at heart and formalise β-effectivity. Thus, CAL has a game-theoretic flavour to it and is reminiscent of coalition logic [39], alternating-time temporal logic [7], and game logic [40].

Providing a sound and complete axiomatisation of CAL is an open problem. Hence we will discuss an extension with common knowledge of a related logic with coalition announcement — *coalition and relativised group announcement logic* (CoRGAL) [27]. Compared to the language of CAL, the language of CoRGAL have additional constructs $[G,\chi]\varphi$ that are called *relativised group announcements*, and that mean 'given true announcement χ, whatever agents from group G announce at the same time, they cannot avoid φ'. The double quantification of CAL modalities seems to be one of the reasons why finding an axiomatisation of CAL is hard. Relativised group announcements allow to split the double quantification and treat coalition's announcements and the anti-coaltion's response separately. In other words, formulas χ within modalities $[G,\chi]\varphi$ act as a kind of memory that stores announcements by a coalition.

Formally, the semantics of coalition modalities and relativised group announcements is as follows:

$$\begin{array}{lll} M_s \models [\langle G\rangle]\varphi & \text{iff} & M_s \models \psi_G \to \langle \psi_G \land \chi_{A\setminus G}\rangle\varphi \text{ for all } \psi_G \in \mathcal{EL}^G \text{ and some } \chi_{A\setminus G} \in \mathcal{EL}^{A\setminus G} \\ M_s \models [G,\chi]\varphi & \text{iff} & M_s \models \chi \land [\psi_G \land \chi]\varphi \text{ for all } \psi_G \in \mathcal{EL}^G. \end{array}$$

The axiomatisation of CoRGAL is an extension of the proof system of PAL with the following axioms and rules of inference:

$$\begin{array}{ll} (A10) & [G,\chi]\varphi \to \chi \land [\psi_G \land \chi]\varphi \text{ for any } \psi_G \in \mathcal{EL}^G \\ (A11) & [\langle G\rangle]\varphi \to \langle A \setminus G, \psi_G\rangle\varphi \text{ for any } \psi_G \in \mathcal{EL}^G \end{array}$$

$$IRG \quad \text{From } \{\eta(\chi \wedge [\psi_G \wedge \chi]\varphi) \mid \psi_G \in \mathcal{EL}^G\}, \text{ infer } \eta([G, \chi]\varphi)$$
$$ICA \quad \text{From } \{\eta(\langle A \setminus G, \psi_G \rangle\varphi) \mid \psi_G \in \mathcal{EL}^G\}, \text{ infer } \eta([\langle G \rangle]\varphi).$$

Extending the proof system with axiom ($A9$) and rule IC from Definition 5.2 results in the axiomatisation of *CoRGAL with common knowledge* (CoRGALC). A similar logic where agents are allowed to make announcements involving common knowledge is denoted with CoRGALC^X, and its proof system can be obtained from the axiomatisation of CoRGALC with the following changes: $[G, \chi]^X\varphi$ replaces $[G, \chi]\varphi$, $[\langle G \rangle]^X\varphi$ replaces $[\langle G \rangle]\varphi$, and $\mathcal{EL}^G$ is substituted by $\mathcal{ELC}^G$, where

$$M_s \models [\langle G \rangle]^X\varphi \quad \text{iff} \quad M_s \models \psi_G \to \langle \psi_G \wedge \chi_{A \setminus G} \rangle\varphi \text{ for all } \psi_G \in \mathcal{ELC}^G \text{ and some } \chi_{A\setminus G} \in \mathcal{ELC}^{A\setminus G}$$
$$M_s \models [G, \chi]^X\varphi \quad \text{iff} \quad M_s \models \chi \wedge [\psi_G \wedge \chi]\varphi \text{ for all } \psi_G \in \mathcal{ELC}^G.$$

Theorem 18. Both CoRGALC and CoRGALC^X are sound and complete.

Completeness in both cases can be shown by combining the corresponding proofs from [27] and Section 5 of the current paper.

6.3 Beyond announcements

Quantification over public announcements is quite well-studied, and one wonders whether similar results could be obtained for other DELs with quantification. As it turns out, quantifying over other modes of information change may yield unexpected results. For example, *action model logic* [20], which allows us to reason about many other types of information changing scenarios apart from public announcement, e.g. private announcements, cheating, gossip, suspicion, etc., once being extended with quantification over action models, is as expressive as epistemic logic [32]. This trivially leads to the fact that such a logic is, for example, decidable.

The fact that action model logic with quantification over action models is as expressive as epistemic logic is due to the existence of so-called 'reduction axioms' that allow one to translate any formula of the former into an equivalent formula of the latter. The same technique has been employed to show completeness of axiomatisations of other logics with quantification over information change, for example refinement modal logic [13] and arbitrary arrow update model logic [22].

However, there also logics with quantification that have only infinitary known axiomatisations. Since, as a rule, completeness proofs for such logics are based on the completeness proof for APAL [10], we can use our proof from Section 5 to deal with the extensions of such logics with common knowledge.

One of such logic is *arbitrary arrow update logic* (AAUL) [21] that extends modal logic K with dynamic arrow updates. Compared to public announcements, arrow updates, as is hinted in the name, focus on arrows rather than states. Informally, an arrow update is a set of triples (φ, a, ψ) that mean that in the updated model a-arrows between φ-states and ψ-states will be preserved. Arrows that do not satisfy any of the triples in the update operator are deleted from a model. Since arrow updates delete arrows, equivalence relations between states are not guaranteed to be preserved, unlike in PAL.

Formally, the language of AAUL extends the language of modal logic K with constructs $[U]\varphi$ and $[\updownarrow]\varphi$. The former means that 'after arrow update U, φ is true', and the latter is read as 'after any arrow update, φ is true'. The semantics of the new operators is as follows:

$$\begin{array}{lll} M_s \models [U]\varphi & \text{iff} & M_s^U \models \varphi \\ M_s \models [\updownarrow]\varphi & \text{iff} & M_s \models [U]\varphi \text{ for each } U \in \mathcal{AUL}, \end{array}$$

where $\mathcal{AUL}$ is a fragment of $\mathcal{AAUL}$ that does not contain $[\updownarrow]$, and $M^U = (S, R^U, V)$ with $R^U(a) = \{R(a)(s,t) \mid \exists(\varphi, a, \psi) \in U : M_s \models \varphi \text{ and } M_t \models \psi\}$. Note that R in M is not necessarily an equivalence.

The reader can see that the axiomatisation of AAUL is quite similar in form to the proof system of APAL (we present only the part that includes the arbitrary arrow update modality):

$$\begin{array}{rl} (A8) & [\updownarrow]\varphi \to [U]\varphi \text{ for any } U \in \mathcal{AUL} \\ R4 & \text{From } \{\eta([U]\varphi) \mid U \in \mathcal{AUL}\}, \text{ infer } \eta([\updownarrow]\varphi). \end{array}$$

Arbitrary arrow update logic with common knowledge (AAULC) was first proposed in [36], where the author showed that the logic is not finitely axiomatisable. The way it was presented, $[\updownarrow]$ quantified over *$\mathcal{AUL}$ with common knowledge*. In order to obtain a proof system for AAULC it is enough to add axiom ($A9$) and rule IC from Section 5 to the axiomatisation of AAULC from [21]. The completeness can be shown by combining the proofs for the completeness of AAUL and GALC, e.g., we would require a theory to be closed under MP, IC, and ($R4$).

Theorem 19. AAULC is sound and complete.

7 Discussion

We studied common knowledge in the context of quantification over information change. In particular, we presented extensions of APAL and GAL with the common knowledge modality, both conservative and with the extended semantics. The extensions are called APALC, APALC^X, GALC, and GALC^X. According to the conservative semantics, the semantics of group and arbitrary announcement modalities is exactly the same as in in APAL and GAL, quantifying over formulas of epistemic logic. Extended semantics allowed group and arbitrary announcements to quantify over a larger set of formulas, namely epistemic logic with common knowledge. We observed that difference in the semantics matters: with the extended semantics we can express properties we cannot express with the conservative semantics, and (perhaps more surprisingly) vice versa. This echoes the results for GAL extended with distributed knowledge [29, 1]. A current expressivity map of $\mathcal{GALC}$, $\mathcal{APALC}$, and other connected logics is shown in Figure 10.

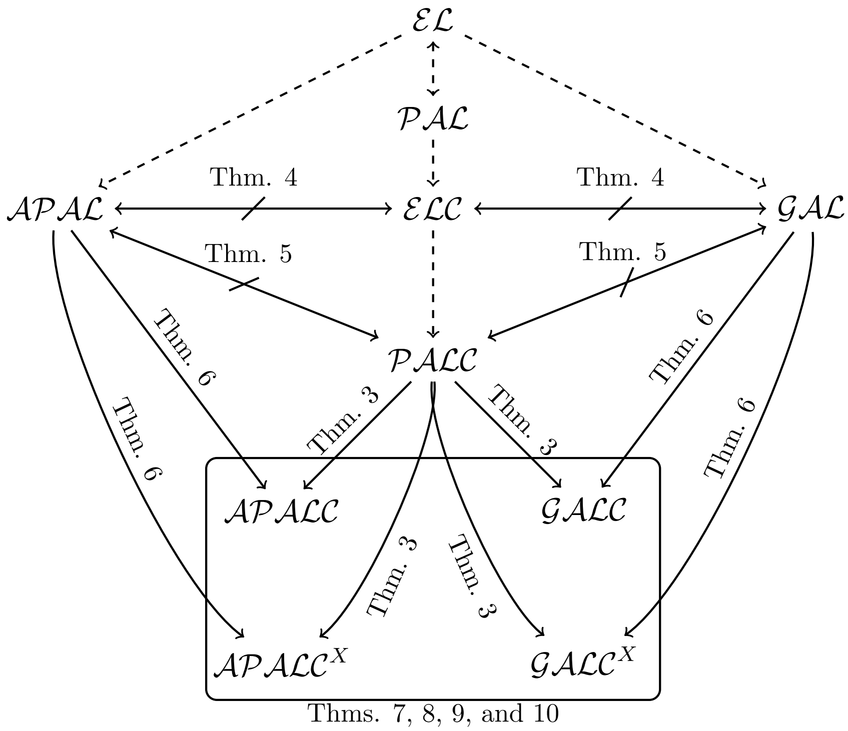

Figure 10: Overview of the expressivity results. An arrow from $\mathcal{L}_1$ to $\mathcal{L}_2$ means $\mathcal{L}_1 \leqslant \mathcal{L}_2$. If there is no symmetric arrow, then $\mathcal{L}_1 < \mathcal{L}_2$. This relation is transitive, and we omit transitive arrows in the figure. An arrow from $\mathcal{L}_1$ to $\mathcal{L}_2$ is crossed-out, if $\mathcal{L}_1 \not\leqslant \mathcal{L}_2$. Dashed arrows depict results known from literature, and solid arrows show the results proven in this paper. All languages in the rounded rectangle are pairwise incomparable.

Moreover, we presented sound and complete axiomatisations of GALC, GALC^X, APALC and APALC^X. We also showed that our proof of the completeness of the axiomatisations can be used to obtain axiomatisations of other logics with quantification over information change and show their completeness.

Throughout the paper we sidestepped one particular fact that deals with public announcements and common knowledge. For the semantics of usual APAL and GAL, there is no difference whether quantification is over formulas of $\mathcal{EL}$ or formulas of $\mathcal{PAL}$. This is a trivial corollary of the fact that both languages are equally expressive [41]. The same, however, cannot be said about the extensions of $\mathcal{EL}$ and $\mathcal{PAL}$ with common knowledge — $\mathcal{ELC}$ and $\mathcal{PALC}$ correspondingly. In particular, $\mathcal{PALC}$ is strictly more expressive than $\mathcal{ELC}$ [20, Theorem 8.48]. Thus, there is yet another way to extend APAL and GAL with common knowledge, i.e. to allow quantification over formulas of $\mathcal{PALC}$. We can call the resulting logics APALC^{XX} and GALC^{XX} with the semantics being as follows:

$$
\begin{aligned}
M_s \models [!]^{XX}\varphi \quad &\text{iff} \quad M_s \models [\psi]\varphi \text{ for all } \psi \in \mathcal{PALC}\\
M_s \models [G]^{XX}\varphi \quad &\text{iff} \quad M_s \models [\psi_G]\varphi \text{ for all } \psi_G \in \mathcal{PALC}^G.
\end{aligned}
$$

While we can yet again reuse our completeness proof to obtain sound and complete axiomatisations of APALC^{XX} and GALC^{XX}, their relative expressivity is left as an open question. Perhaps more intriguing open problem is specifying the exact relationship between APALC^{XX} and APALC^{X}, and GALC^{XX} and GALC^{X}.

In the same vein, it is worthwhile to investigate expressivities of other logic with quantification over information change mentioned in the article, e.g., coalition announcement logic with common knowledge, positive APAL with common knowledge, or arbitrary arrow update logic with common knowledge.

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