# Group Knowledge in Public Communication

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# First things first

Based on the upcoming

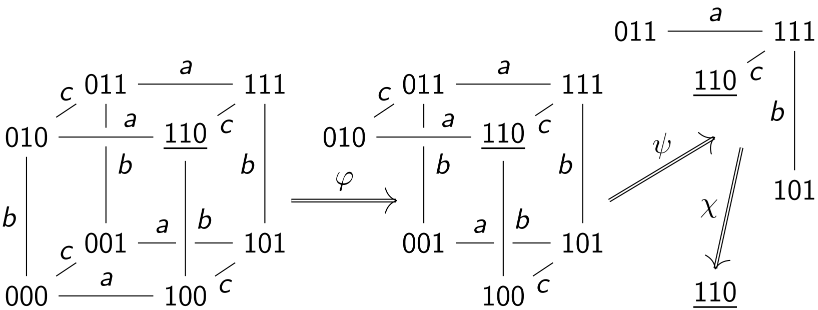
- Quantified Announcements and Common Knowledge. In Proceedings of the 20th AAMAS (2021). (with Thomas Ågotnes)

# PAL: Muddy Children

### Muddy Children

Three children — Alice ($a$), Bobby ($b$), and Claire ($c$) — have been playing outside, and some of them may have mud on their foreheads. Each child can see every other child's forehead, but not their own... Then, their parent says: **'At least one of you has mud on their forehead'**, and, after that, **'If you know if you have a muddy forehead, please step forward'**. If nobody steps forward, the parent keeps repeating the request.

'At least one child is muddy': $\varphi := m_a \vee m_b \vee m_c$
'Nobody steps forward': $\psi := \bigwedge_{i \in \{a,b,c\}} \neg(\Box_i m_i \vee \Box_i \neg m_i)$
'Alice and Bobby step forward': $\chi := \Box_a m_a \wedge \Box_b m_b$



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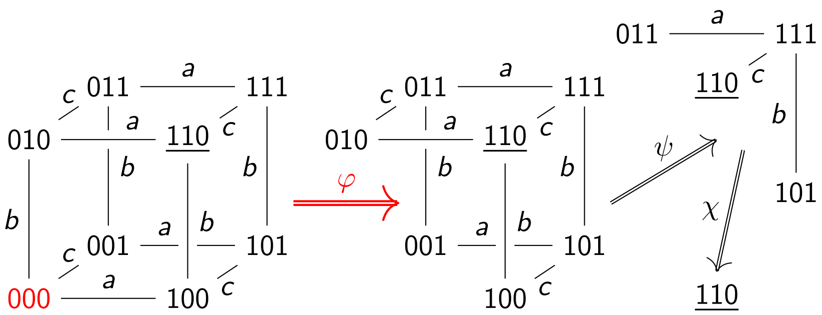

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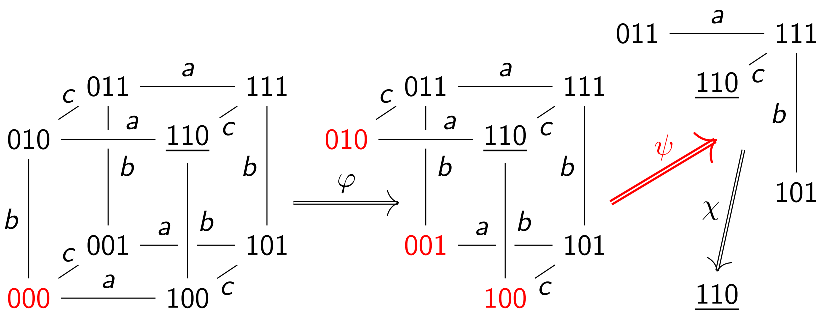

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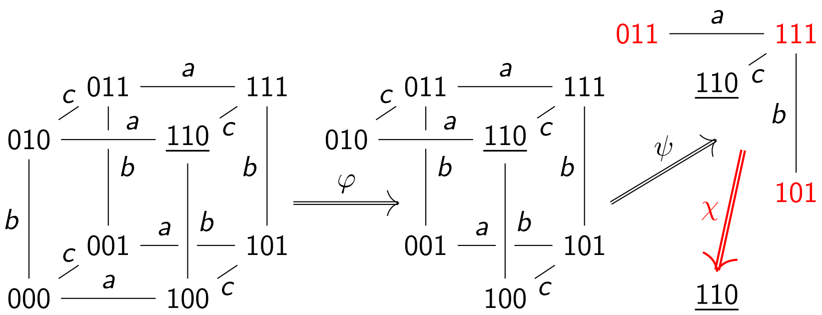


# PAL: Syntax and Semantics

$$\mathcal{PAL} \ni \varphi ::= p \mid \neg\varphi \mid \varphi \land \varphi \mid \Box_a\varphi \mid [\varphi]\varphi$$

**Definition (Semantics)**

An announcement of $\varphi$ in a pointed model $M_s$ results in an updated pointed model $M_s^\varphi$ containing only $\varphi$-states:

- $S^\varphi = [\![\varphi]\!]_M$,
- $\sim_a^\varphi = \sim_a \cap (S^\varphi \times S^\varphi)$,
- $V^\varphi(p) = V(p) \cap S^\varphi$.

$$M_s \models [\varphi]\psi \quad \text{iff} \quad M_s \models \varphi \text{ implies } M_s^\varphi \models \psi$$
$$M_s \models \langle\varphi\rangle\psi \quad \text{iff} \quad M_s \models \varphi \text{ and } M_s^\varphi \models \psi$$

# PAL: Properties

For each $\varphi \in \mathcal{PAL}$ there is an equivalent $t(\psi) \in \mathcal{EL}$, where $t$ is a translation function.

**Theorem**

*PAL is sound and complete.*

**Theorem**

*PAL and EL are equally expressive.*

# GAL: Muddy Children

## Strategic Muddy Children

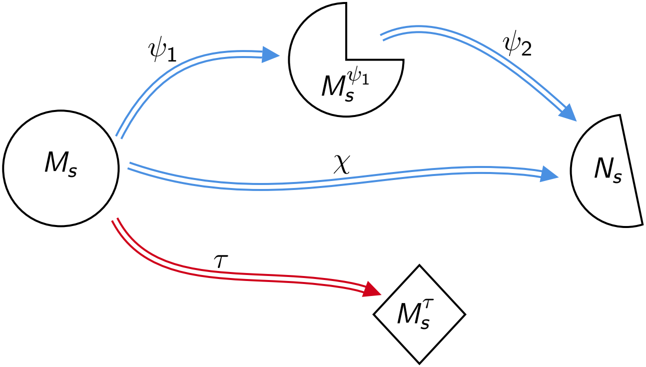
Alice, Bobby, and Claire may not need a parent to learn who is muddy. **Can they communicate so that all of them know exactly who is muddy?** Of course. E.g. Claire can say that Alice and Bobby are muddy, and Alice can say that Clair's forehead is not muddy.

**Can Claire let Alice know that her forehead is muddy without Bobby knowing about the interaction?** No, not in the context of public communication.

# GAL: Muddy Children

## Strategic Muddy Children

More generally, having an initial model $M_s$ is there an announcement (or a sequence thereof) by a group (or groups) of agents such that some goal model $N_s$ is reachable?



# GAL: Syntax and Semantics

Group Announcement Logic (GAL) = PAL + $\{[G]\varphi, \langle G\rangle\varphi\}$

$\langle G\rangle\varphi$: 'agents from $G$ have a joint announcement such that $\varphi$ holds in the resulting model'

$[G]\varphi$: 'whatever agents from $G$ announce, they cannot avoid $\varphi$'

Let $\psi_G := \bigwedge_{a\in G} \Box_a \psi_a$, where $\psi_a$ is an epistemic formula (truthfulness).

**Definition (Semantics)**

$$M_s \models [G]\varphi \quad \text{iff} \quad \forall \psi_G : M_s \models [\psi_G]\varphi$$
$$M_s \models \langle G\rangle\varphi \quad \text{iff} \quad \exists \psi_G : M_s \models \langle \psi_G\rangle\varphi$$

# GAL: Properties

| | |
|---|---|
| PAL | $[G]\varphi \to [\psi_G]\varphi$ |
| From $\forall \psi_G : \eta([\psi_G]\varphi)$ infer $\eta([G]\varphi)$ | From $\varphi$ infer $[G]\varphi$ |

The axiomatisation is infinitary

**Theorem**

*GAL is sound and complete.*

**Properties**

$\langle G\rangle\langle G\rangle\varphi \leftrightarrow \langle G\rangle\varphi$ A multi-step strategy is reducible to a single-step strategy.
$\langle G \cup H\rangle\varphi \not\to \langle G\rangle\langle H\rangle\varphi$ It is not the case for different groups
$\langle G\rangle[H]\varphi \not\to [H]\langle G\rangle\varphi$ Confluence does not hold

# Why Group Knowledge?

Formalising the notion of ability: if agents can reach a configuration of a model, should they be aware it? Everyone? Should it be a common\distributed knowledge?

More complex epistemic goals: it is common knowledge among agents in $G$ that no one in $H$ knows $\varphi$; agents in $G$ should all implicitly know $\varphi$ but this should not be known explicitly.

Preconditions for different types of agents: a shy agent on a social network will publicly show support for cause $\varphi$, only if they know that it is common knowledge in their friend group that everyone supports $\varphi$.

# Notions of Group Knowledge

Multiple acting agents and their knowledge $\Rightarrow$ group knowledge

## Everybody knows

'Everybody knows that $\varphi$' is expressible in GAL.

$$\Box_G \varphi := \bigwedge_{i \in G} \Box_i \varphi$$

## Distributed knowledge

'If agents share their knowledge, then they will know $\varphi$'. $D_G \varphi$ is not expressible in GAL.

$$\sim_G = \bigcap_{i \in G} \sim_i$$

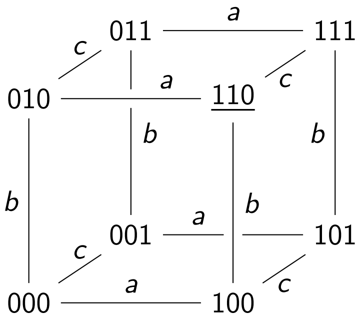
# Common Knowledge

It is common knowledge that $\varphi$ iff everyone knows $\varphi$, everyone knows that everyone knows $\varphi$, and so on.

**The same thing a bit more formally**

$\blacksquare_G \varphi := \bigwedge_n \Box_G^n \varphi$ is not expressible in GAL (infinite formula).

$$M_s \models \blacksquare_G \varphi \quad \text{iff} \quad \forall n \in \mathbb{N} : M_s \models \Box_G^n \varphi$$




$M_{110} \models \blacksquare_a m_b$

$M_{110} \models \blacksquare_{\{a,c\}}(m_b \wedge \neg \Box_a m_b)$

$M_{110} \not\models \blacksquare_{\{a,b,c\}}(m_a \vee m_b \vee m_c)$

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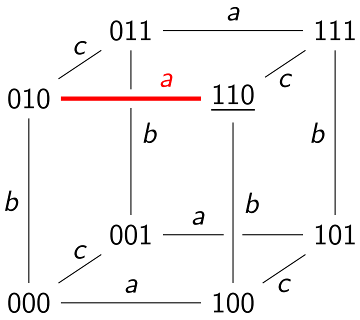


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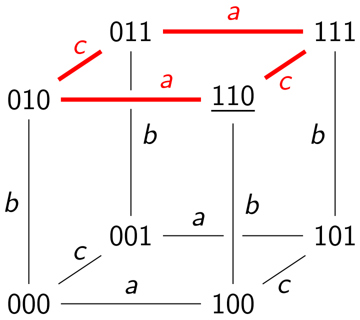


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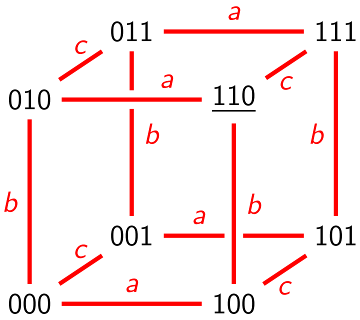


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# Group Knowledge + Group Communication

GAL with Common Knowledge (GALC) = GAL + $\blacksquare_G \varphi$

Let $\psi_G := \bigwedge_{a \in G} \Box_a \psi_a$, where $\psi_a$ is an epistemic formula.

**Definition (Semantics)**

$$M_s \models \blacksquare_G \varphi \quad \text{iff} \quad \forall n \in \mathbb{N} : M_s \models \Box_G^n \varphi$$
$$M_s \models [G]\varphi \quad \text{iff} \quad \forall \psi_G : M_s \models [\psi_G]\varphi$$

However, there is a subtlety...

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However, there is a subtlety... With the extended base language, we can also allow agents to announce formulas with $\blacksquare_G$ (still no circularity).

# Group Knowledge + Group Communication

GAL with Common Knowledge (GALC) = GAL + $\blacksquare_G \varphi$
Let $\psi_G := \bigwedge_{a \in G} \Box_a \psi_a$, where $\psi_a$ is an **epistemic** formula.

## Definition (Semantics)

$$M_s \vDash \blacksquare_G \varphi \quad \text{iff} \quad \forall n \in \mathbb{N} : M_s \vDash \Box^n_G \varphi$$
$$M_s \vDash [G]\varphi \quad \text{iff} \quad \forall \psi_G : M_s \vDash [\psi_G]\varphi$$

GALC eXtended ($GALC^X$) = PAL + $\{\blacksquare_G \varphi, [G]^X \varphi, \langle G \rangle^X \varphi\}$
Let $\psi_G := \bigwedge_{a \in G} \Box_a \psi_a$, where $\psi_a$ is an **EL+C** formula.

## Definition (Semantics)

$$M_s \vDash [G]^X \varphi \quad \text{iff} \quad \forall \psi_G : M_s \vDash [\psi_G]\varphi$$

# GALC: Properties

### General Case

$\langle G \rangle \varphi \to \langle G \rangle^X \varphi$ If a goal is achieved by simple means, it can be achieved by more complex means

$\Box_G \varphi \not\to \langle G \rangle \blacksquare_H \varphi$ Group knowledge cannot be made common...

$\Box_G \varphi \not\to \langle G \rangle \blacksquare_G \varphi$ ...even within the same group

$\blacksquare_G \varphi \not\to \langle G \rangle \blacksquare_H \varphi$ One cannot freely share common knowledge

Not all knowledge can be shared. The (in)famous offender $p \wedge \neg \Box_a p$ makes knowledge 'unstable'. What if we restrict our attention to 'stable' knowledge?

# GALC: Properties

Positive (Universal) Fragment of ELC = box modalities + only propositions are negated
We cannot express that someone does not know something.

**All $\varphi$'s are positive**

$\Box_G \varphi \to \langle G \rangle \blacksquare_H \varphi$
$\Box_G \varphi \to \langle G \rangle \blacksquare_G \varphi$
$\blacksquare_G \varphi \to \langle G \rangle \blacksquare_H \varphi$

# GALC: Axiomatisation

| | |
|---|---|
| PAL | $[G]\varphi \to [\psi_G]\varphi$ |
| From $\forall \psi_G : \eta([\psi_G]\varphi)$ infer $\eta([G]\varphi)$ | From $\varphi$ infer $[G]\varphi$ |
| From $\forall n \in \mathbb{N} : \eta(\Box^n_G \varphi)$ infer $\eta(\blacksquare_G \varphi)$ | $\blacksquare_G \varphi \to \Box^n_G \varphi$ |

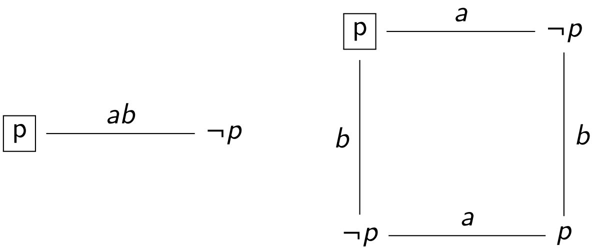
The axiomatisation is infinitary (as GAL). We do not rely on fixed-point axioms for common knowledge

**Theorem**

*GALC is sound and complete.*

# GALC: Bisimulation

The following models are bisimilar



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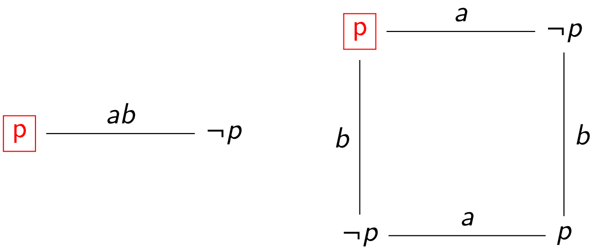

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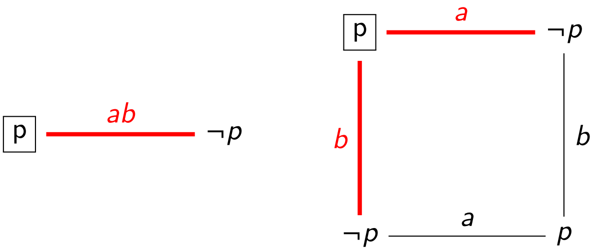

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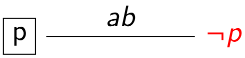


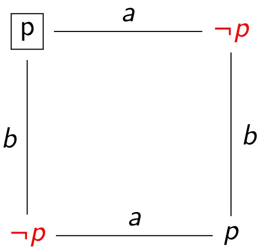

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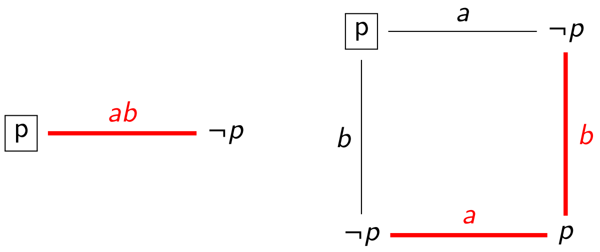

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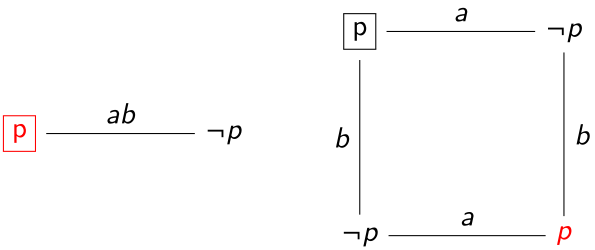

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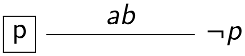


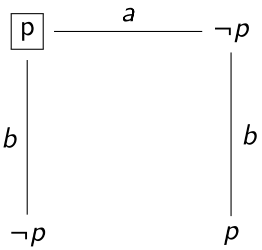

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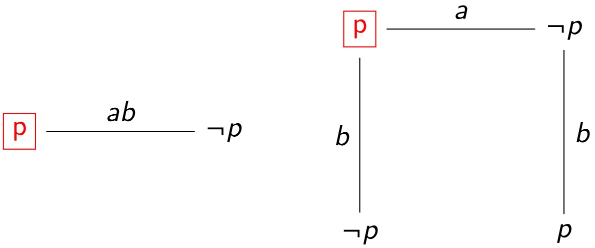

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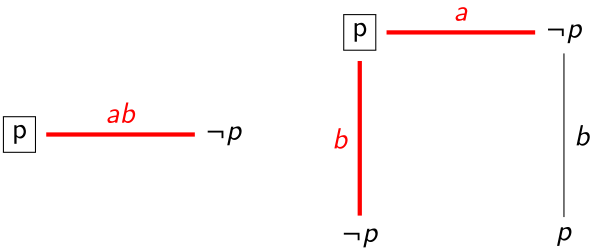

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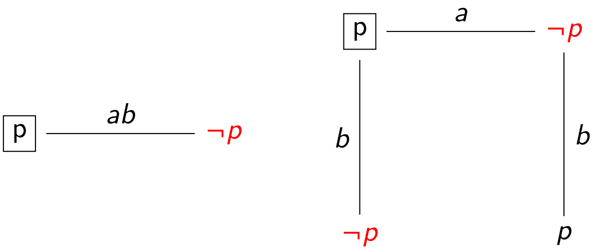

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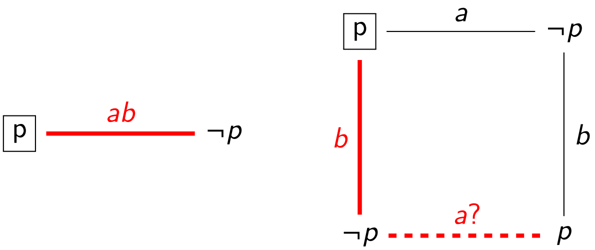

# GALC: Bisimulation

**Theorem**

*Public announcements preserve bisimulation*

**Theorem**

*Bisimilar models satisfy the same formulas of GALC*

**Theorem**

*If bisimilarity between models can be maintained up to $n$ rounds, then the models satisfy the same epistemic formulas up to depth $n$*

# GALC: Bisimulation

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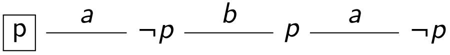


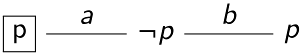


- 2-bisimilar, and not 3-bisimilar; check $\Diamond_a \Diamond_b \Box_a p$
- not bisimilar $\Rightarrow$ dist. by ELC; check $\blacksquare_{\{a,b\}} \Diamond_b p$

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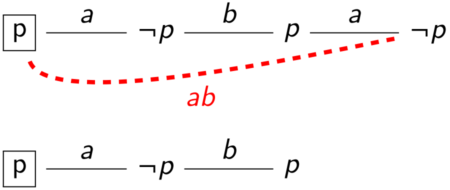


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# GALC: Expressivity

A (bit ugly) reminder...

$$
\begin{array}{lll}
\mathcal{EL} & \ni & \varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_a \varphi \\
\mathcal{ELC} & \ni & \varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_a \varphi \mid \blacksquare_G \varphi \\
\mathcal{PALC} & \ni & \varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_a \varphi \mid \blacksquare_G \varphi \mid [\varphi]\varphi \\
\mathcal{GAL} & \ni & \varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_a \varphi \mid [\varphi]\varphi \mid [G]\varphi \\
\mathcal{GALC} & \ni & \varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_a \varphi \mid \blacksquare_G \varphi \mid [\varphi]\varphi \mid [G]\varphi \\
\mathcal{GALC}^X & \ni & \varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_a \varphi \mid \blacksquare_G \varphi \mid [\varphi]\varphi \mid [G]^X \varphi
\end{array}
$$

Some preliminary results...

**Theorem**

*Both pairs ELC and GAL, and PALC and GAL are incomparable.*
*GALC and $GALC^X$ are strictly more expressive than GAL.*

# GALC: Expressivity

What about GALC versus $\text{GALC}^X$?

**General Intuition**

Provide two non-bisimilar models that agree on all modalities — $\Box_a\varphi$, $[\psi]\varphi$, $\blacksquare_G\varphi$, $[G]\varphi$— apart from $[G]^X\varphi$.

First, we unpack $[G]^X\varphi$:

$$M_s \models [G]^X\varphi \quad \text{iff} \quad \forall \psi_G \in \mathcal{ELC}^G : M_s \models [\psi_G]\varphi$$

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Let us unpack $[G]^X\varphi$:

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We need the models to differ in some way on some $\psi_G \in \mathcal{ELC}^G$.
But what about their agreement on modalities $\blacksquare_G\varphi$?

# GALC: Expressivity

We show that GALC is not at least as expressive as GALC$^X$. Or, in English, there are some (classes of) models that can be distinguished by a GALC$^X$ formula and not by any GALC formula.

**Proof Idea**

We consider a specific $\varphi \in \mathcal{GALC}^X$ and assume towards a contradiction that there is an equivalent $\psi \in \mathcal{GALC}$. We need to construct two models, $M_s$ and $N_t$, such that $M_s \vDash \varphi$ and $N_t \nvDash \varphi$, and, at the same time, $M_s \vDash \psi$ iff $N_t \vDash \psi$.

Let us consider the intuitions guiding our constructions.

# GALC: Expressivity

We have $\varphi \in \mathcal{GALC}^X$, and assumed that there is an equivalent $\psi \in \mathcal{GALC}$.

## The Intuitions

- Models are *n*-bisimilar for any $n$, and thus agree on all formulas of EL.
- Hence, they agree on all $[G]\chi$ (recall $\psi_G \in \mathcal{EL}$).
- Models are distinguished by some $\blacksquare_G \chi$...

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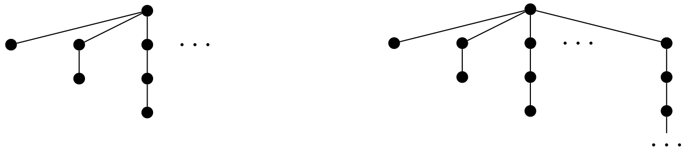
We have $\varphi \in \mathcal{GALC}^X$, and assumed that there is an equivalent $\psi \in \mathcal{GALC}$.

## The Intuitions

- Models are *n*-bisimilar for any *n*, and thus agree on all formulas of EL.
- Hence, they agree on all $[G]\chi$ (recall $\psi_G \in \mathcal{EL}$).
- Models are distinguished by some $\blacksquare_G\chi$...

We can use the classic intuition...



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- Formula $\psi$ is finite $\Rightarrow$ there is some $q \notin Prop(\varphi) \cup Prop(\psi)$.
- Quantification in $[G]^X \chi$ is implicit $\Rightarrow$ we can use formulas with $\blacksquare_G$ and $q$ in the quantification.

# GALC: Expressivity

Combining the intuitions together, we can show that there are some (properties of) models that can be detected by $\text{GALC}^X$ and cannot be detected by GALC.

**Conjecture**

There are (classes of) models that can be distinguished by GALC, and not by $\text{GALC}^X$.

# GAL with Distributed Knowledge

What about distributed knowledge?

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- Group Announcement Logic with Distributed Knowledge. In Proceedings of the 7th LORI (2019), pp. 98–111. Springer. (with Thomas Ågotnes and Natasha Alechina)
- Logics with Group Announcements and Distributed Knowledge: Completeness and Expressive Power. JoLLI (with Thomas Ågotnes and Natasha Alechina; accepted)

# GALC: Overview

- GAL allows to reason about the existence of a public announcement that reaches certain epistemic goals
- GAL was extended with a classic group knowledge modality $\blacksquare_G \varphi$ (GALC)
- Two possible interpretations of the semantics of group announcements: classical and extended (GALC$^X$)
- Axiomatisations of GALC and GALC$^X$ are sound and complete
- GALC and GALC$^X$ are strictly more expressive than GAL. And GALC is not at least as expressive as GALC$^X$
- The same results hold for APAL
- Full expressivity analysis?
- Beyond announcements?