Group Knowledge

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What is Group Knowledge?

Everybody knows. Everybody in G knows φ if each member of the group knows φ

Common knowledge. Everybody in G knows φ and everybody in G knows that everybody in G knows φ, and so on

Distributed knowledge. If agents in G would pool their knowledge together, they would know φ

Van Ditmarsch, Van der Hoek, Kooi. *Dynamic Epistemic Logic*, Section 2. 2008.

Everybody Knows

Everybody knows. Everybody in G knows φ if each member of the group knows φ

$$E_G\varphi := \bigwedge_{a\in G} \Box_a \varphi$$

Semantics

$$M, s \models E_G\varphi \text{ iff } M, s \models \bigwedge_{a\in G} \Box_a \varphi$$

A logic with everybody knows is as expressive as the basic epistemic logic

Van Ditmarsch, Van der Hoek, Kooi. *Dynamic Epistemic Logic*, Section 2. 2008.

Common knowledge

Common knowledge. Everybody in G knows φ and everybody in G knows that everybody in G knows φ, and so on

$$E^0_G\varphi := \varphi$$

$$E^{n+1}_G\varphi := E_G E^n_G\varphi$$

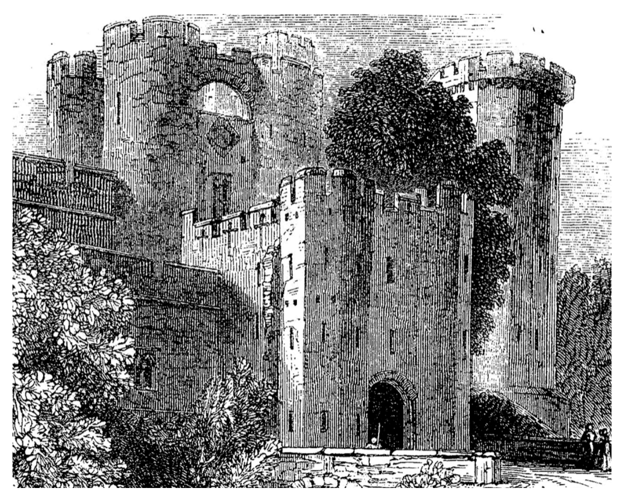
$$C_G\varphi := \bigwedge_{n\in\mathbb{N}} E^n_G\varphi$$

Common knowledge is closely related to the notion of consensus

Van Ditmarsch, Van der Hoek, Kooi. *Dynamic Epistemic Logic*, Section 2. 2008.

Two generals problem

A



C

B

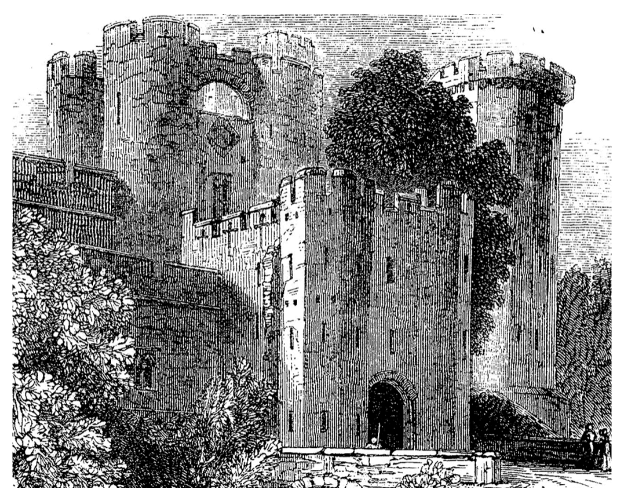
If only one general attacks, they will lose

If two generals attack at the same time, they will capture the castle

Two generals problem

A

B

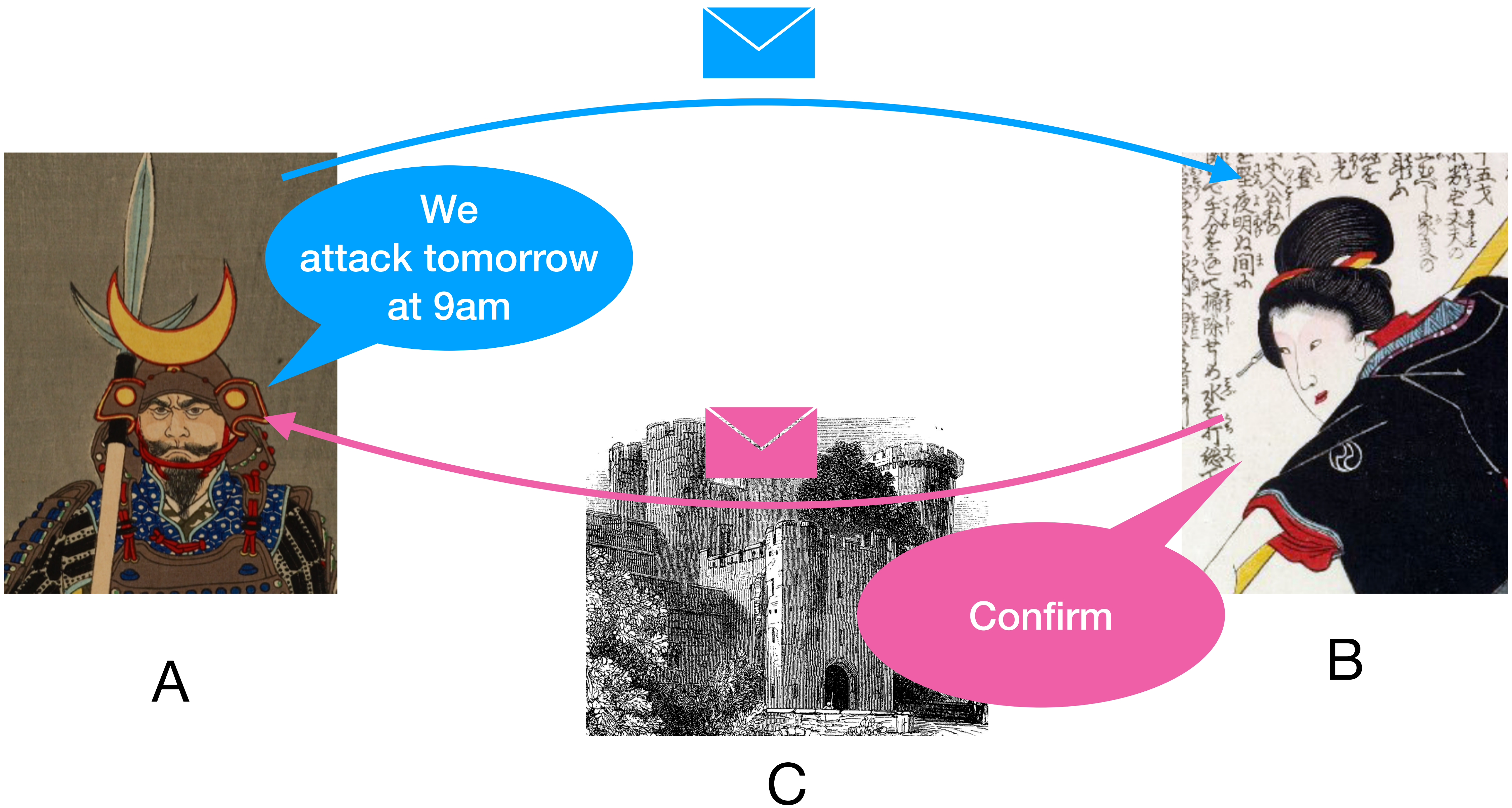


C

Messenger can be captured on their way between the generals

Is sending a message enough?

Two generals problem



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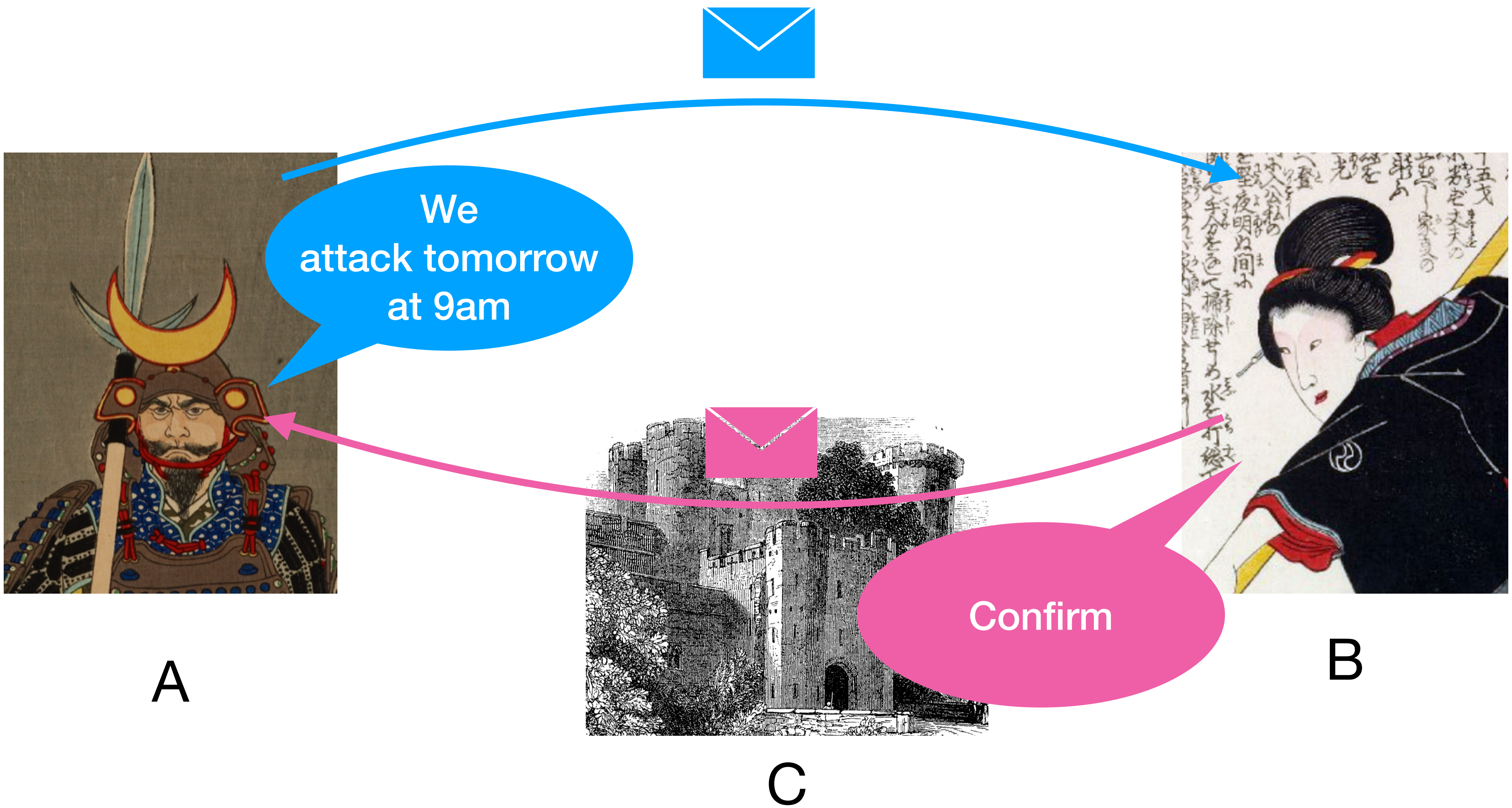

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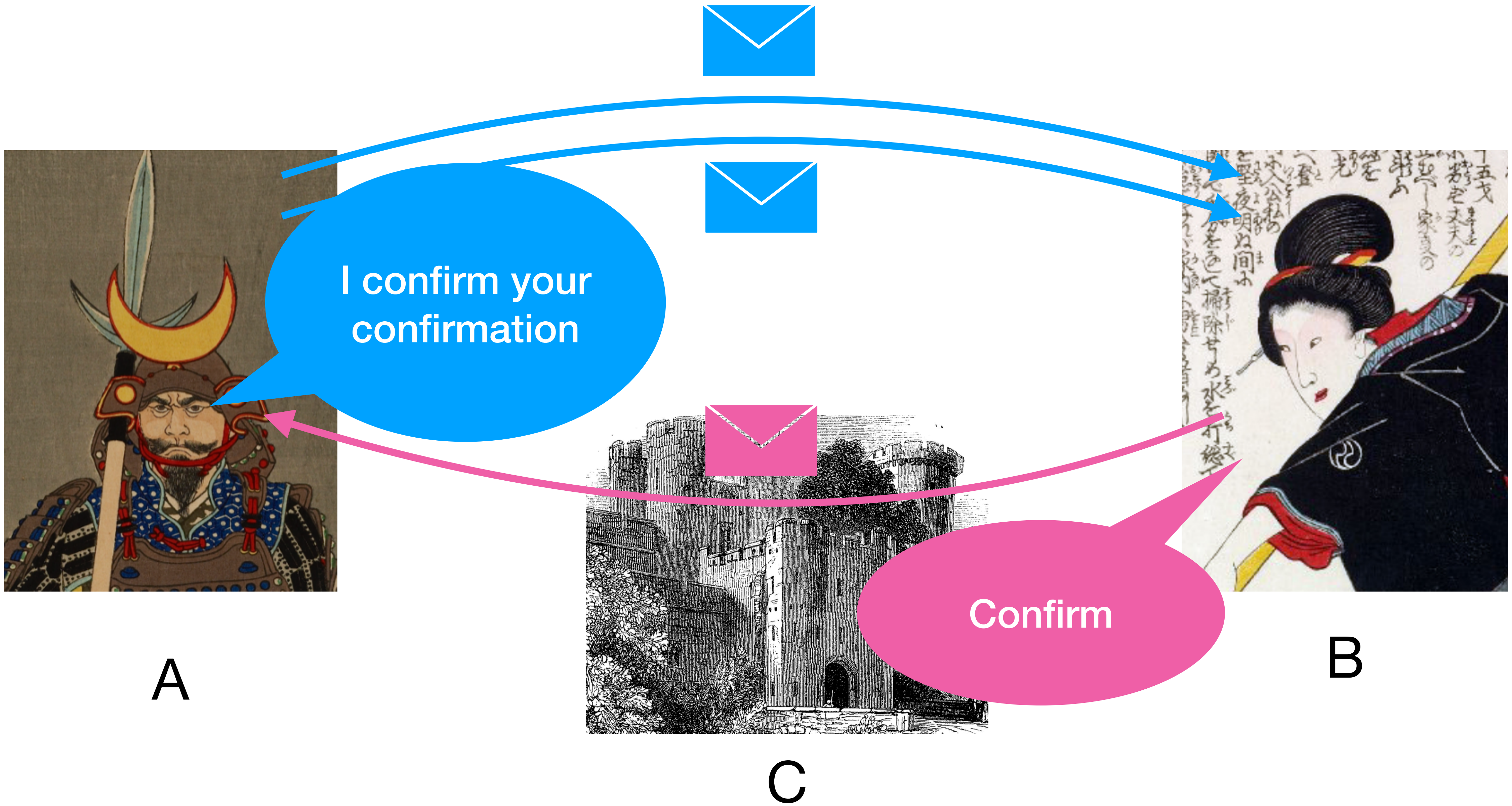
Are the generals ready to attack now?

Two generals problem



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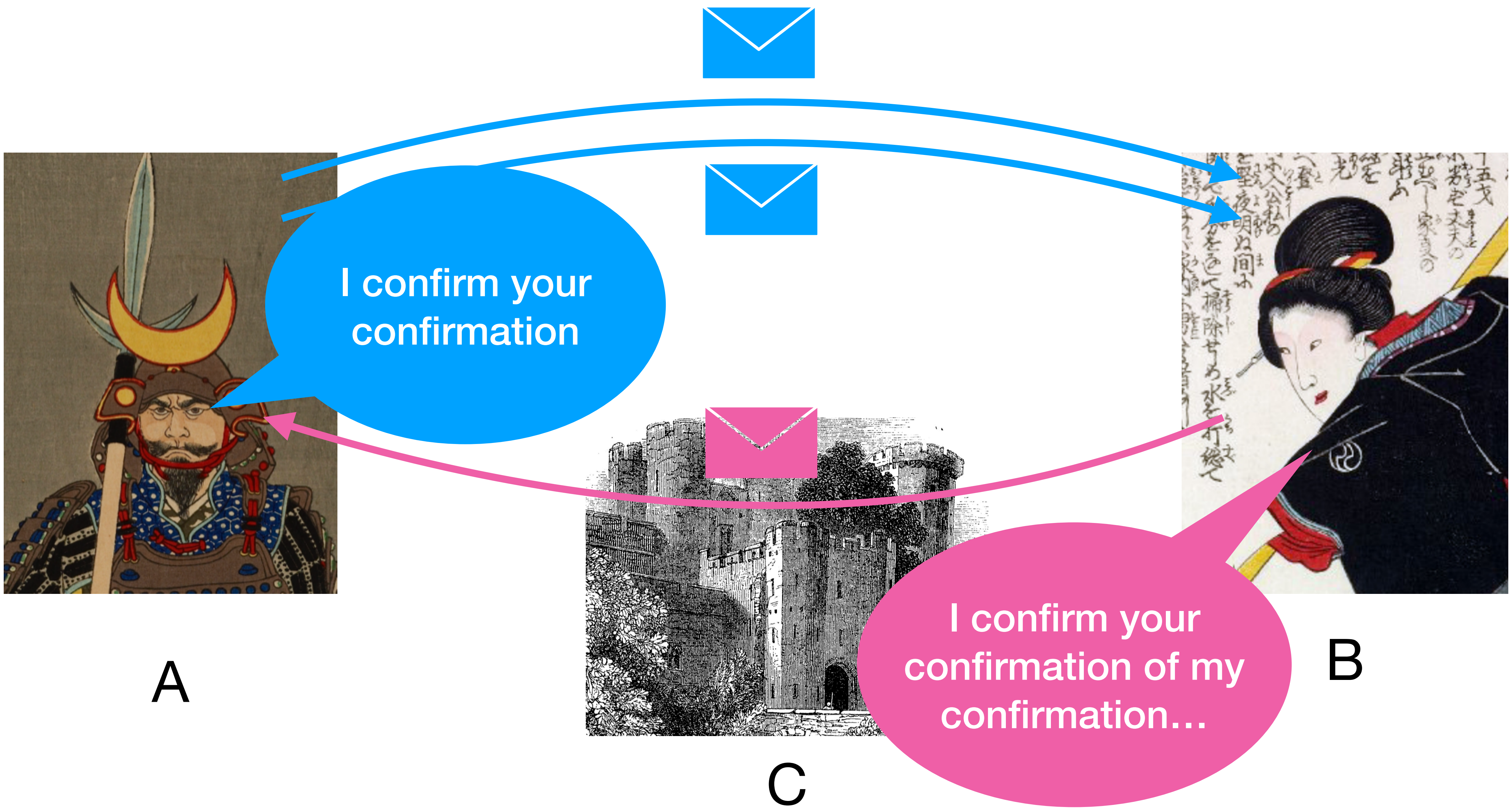

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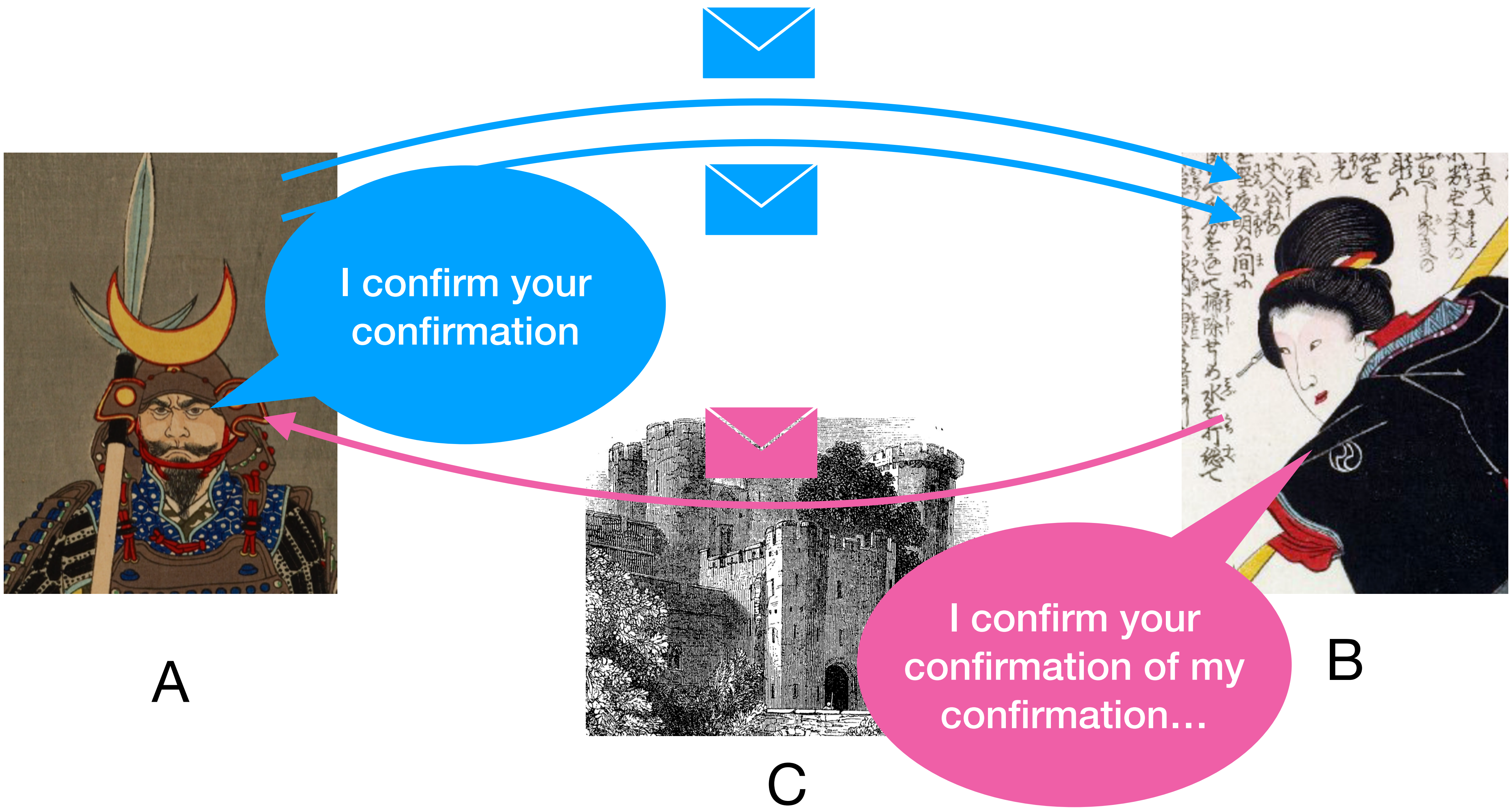

The generals will never reach consensus

Or, the attack time will never become common knowledge

Common knowledge

Common knowledge. Everybody in G knows φ and everybody in G knows that everybody in G knows φ, and so on

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$$E_G^{n+1}\varphi := E_G E_G^n\varphi$$

$$C_G\varphi := \bigwedge_{n\in\mathbb{N}} E_G^n\varphi$$

$$M, s \models C_G\varphi \text{ iff } \forall n \in \mathbb{N} : M, s \models E_G^n\varphi$$
$$M, s \models C_G\varphi \text{ iff } \forall t \in S : s \sim_G^* t \text{ implies } M, t \models \varphi$$

$$\sim_G^* = (\bigcup_{a\in G} \sim_a)^*$$

Equivalent definitions!

Distributed knowledge

Distributed knowledge. If agents in G would pool their knowledge together, they would know φ

Example

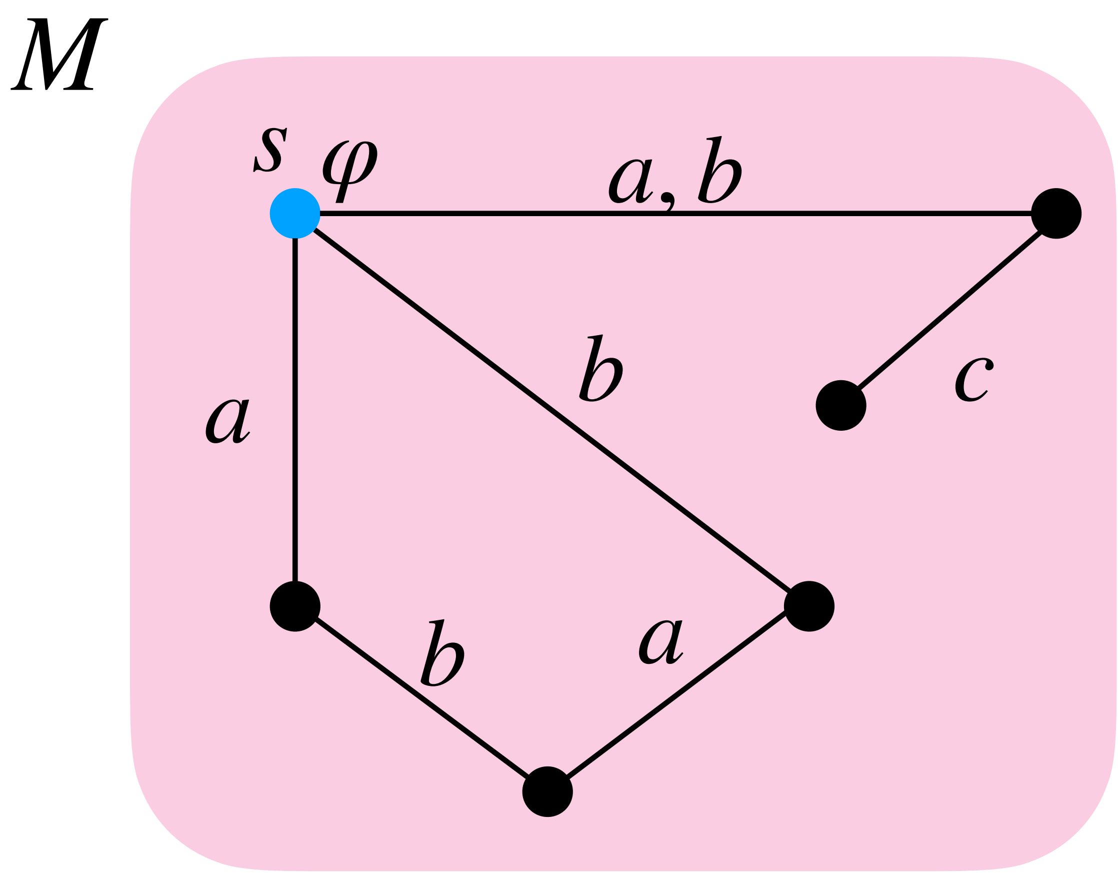
$\Box_a(\varphi \to \psi)$ $\Box_b \varphi$

$D_{\{a,b\}}\psi$

$$\sim_G^{\cap} = \bigcap_{a \in G} \sim_a$$

$M, s \models D_G\varphi$ iff $\forall t \in S : s \sim_G^{\cap} t$ implies $M, t \models \varphi$

Group Knowledge



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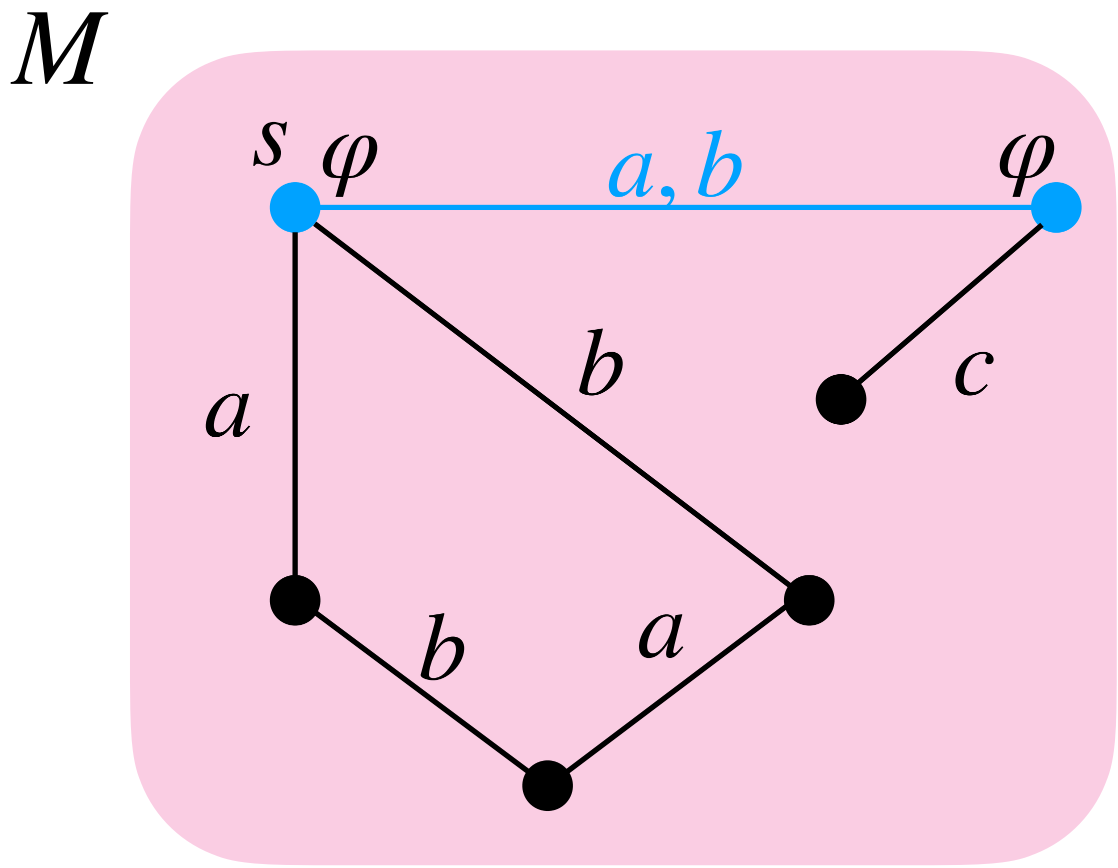

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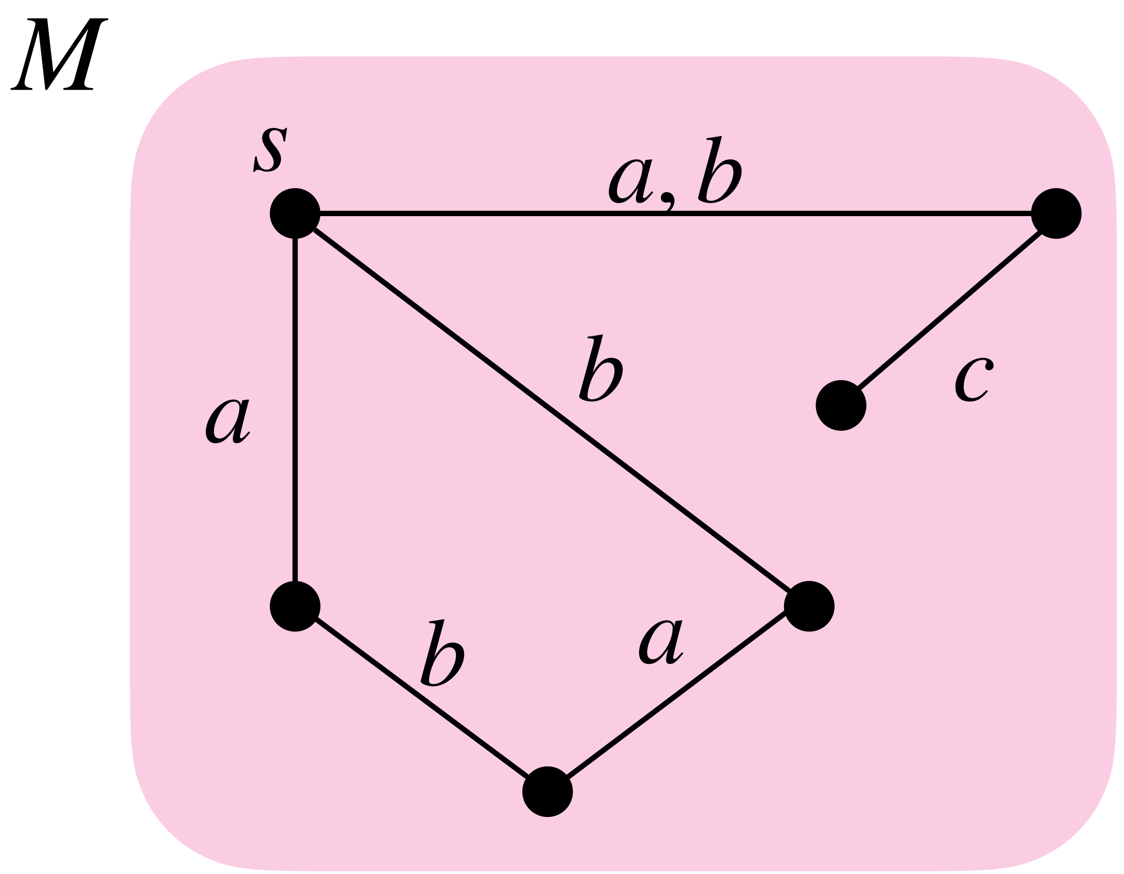

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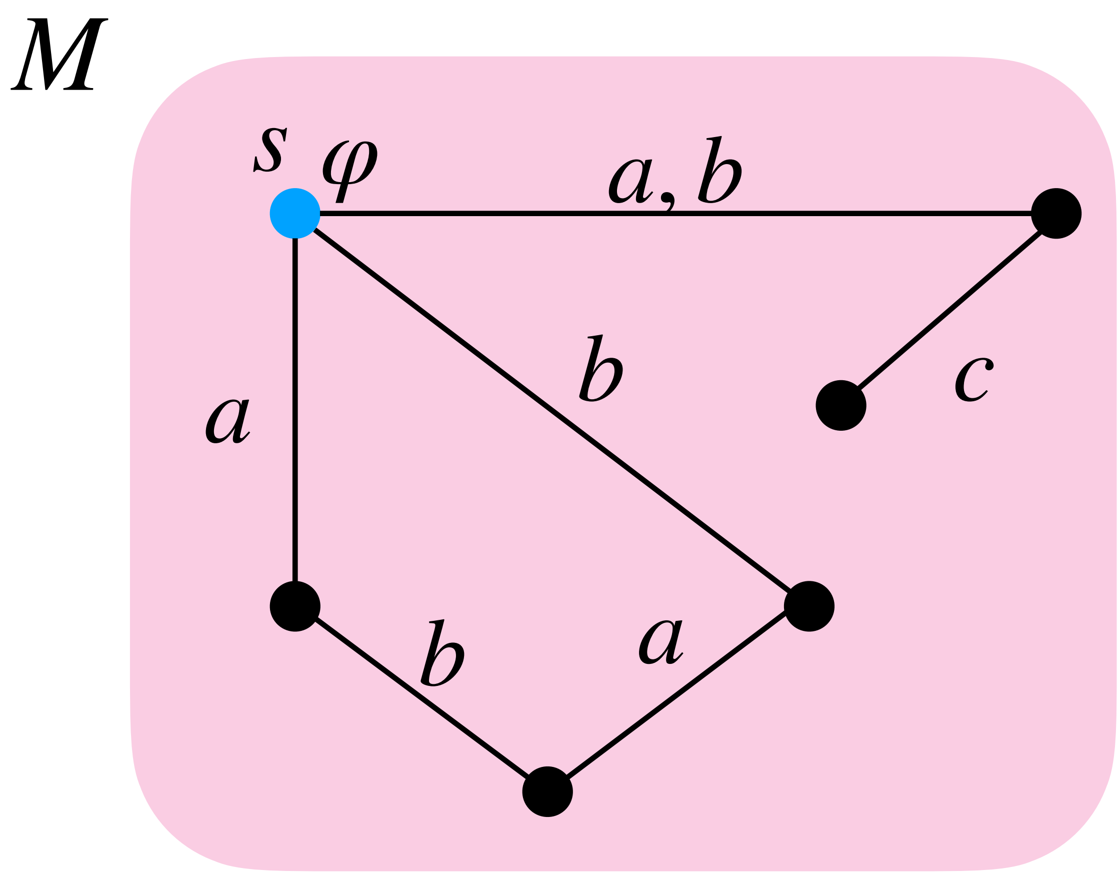

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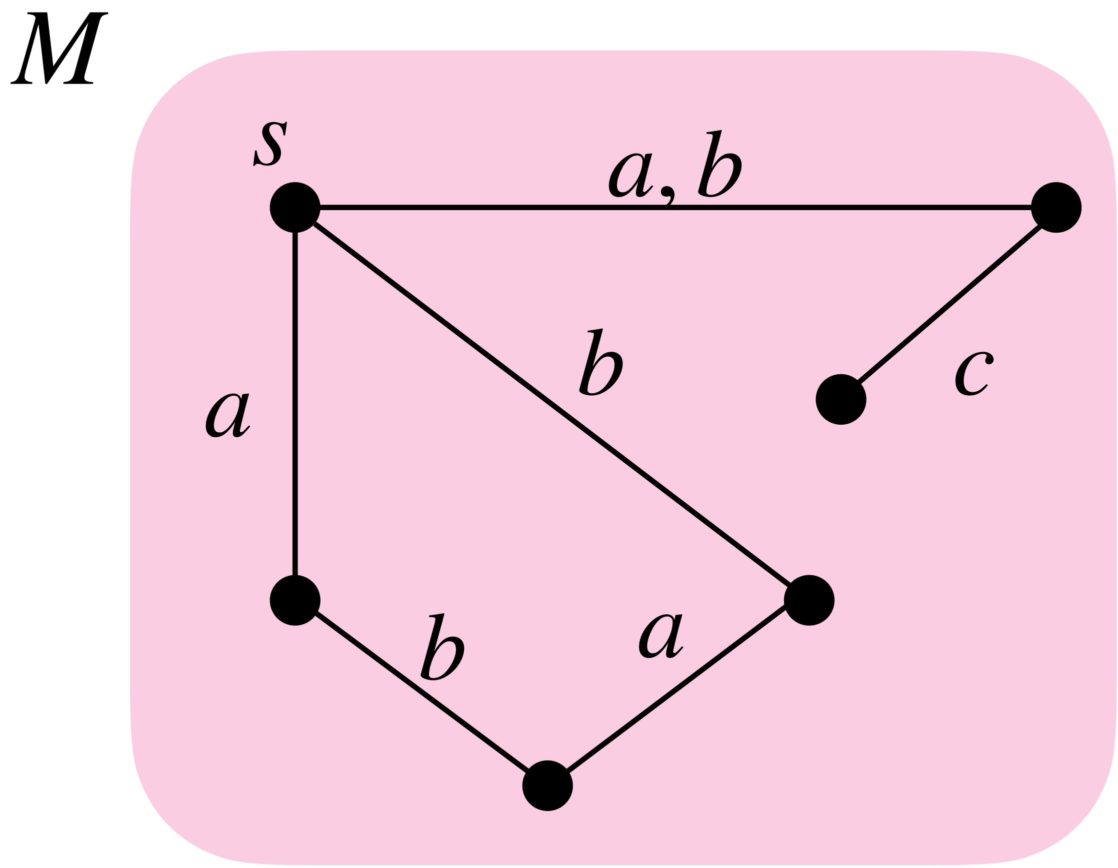

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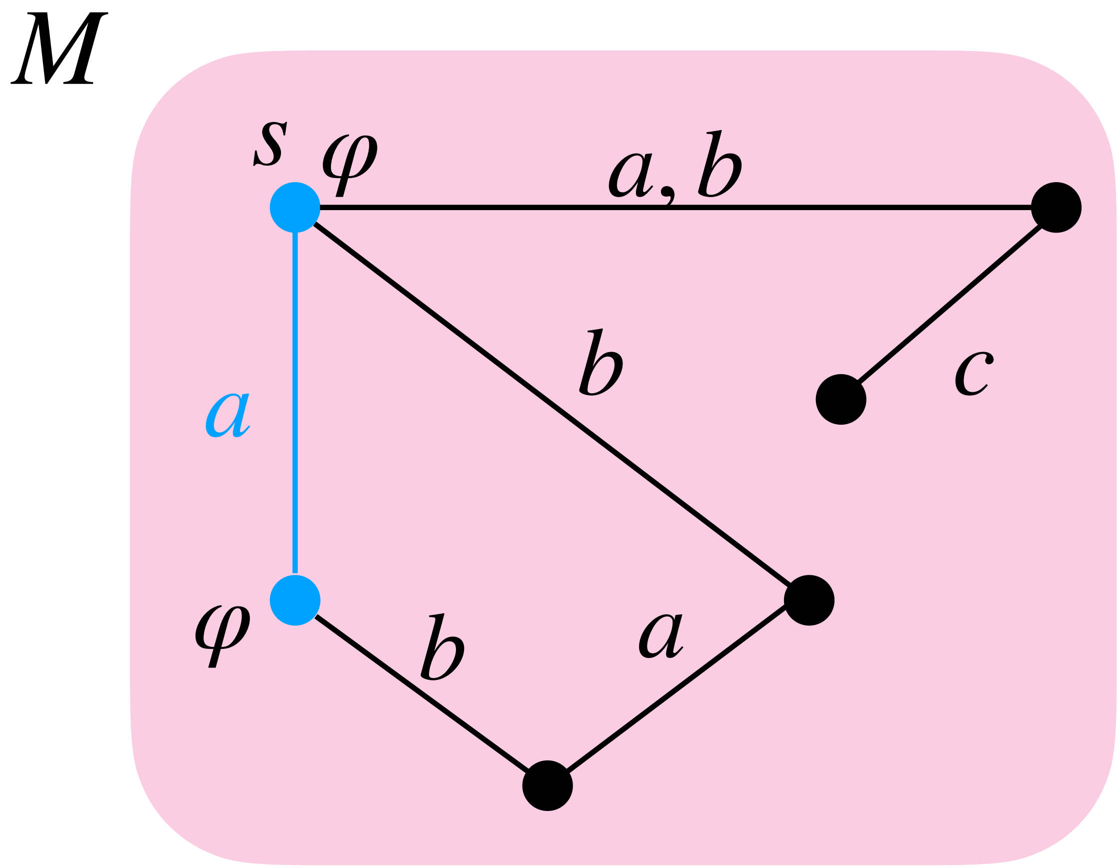

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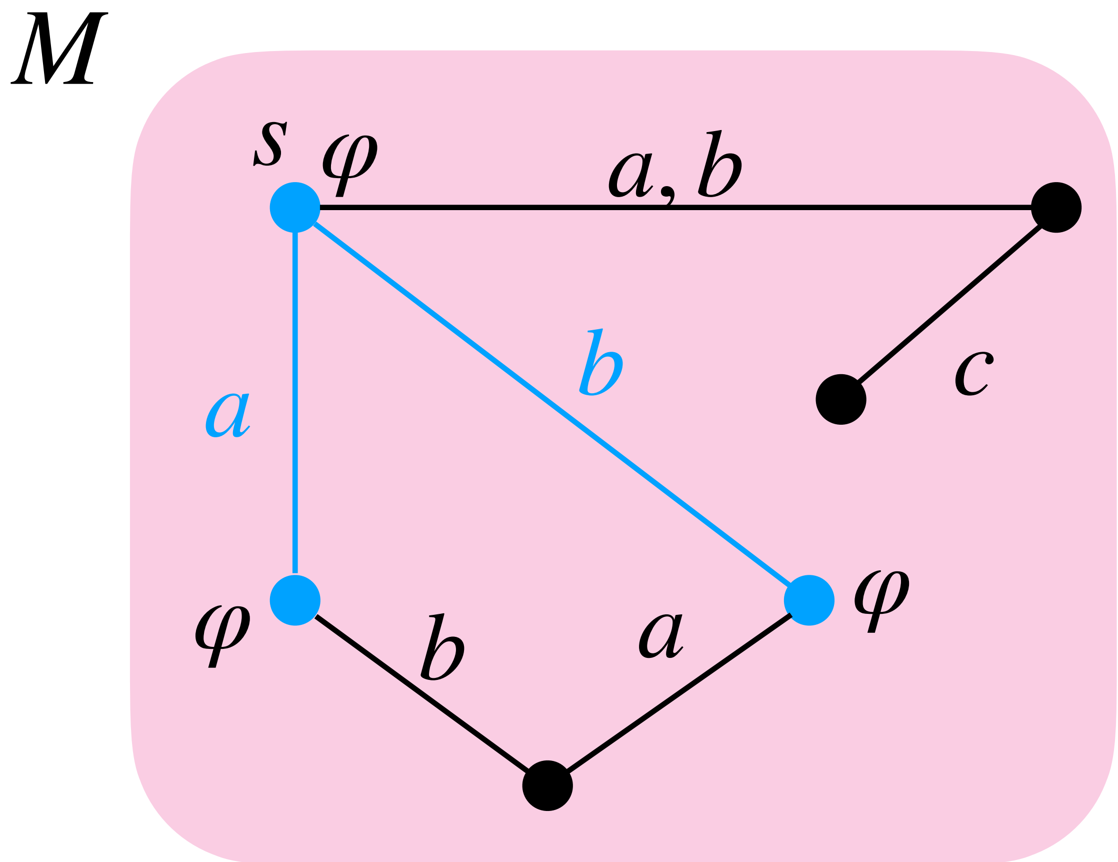

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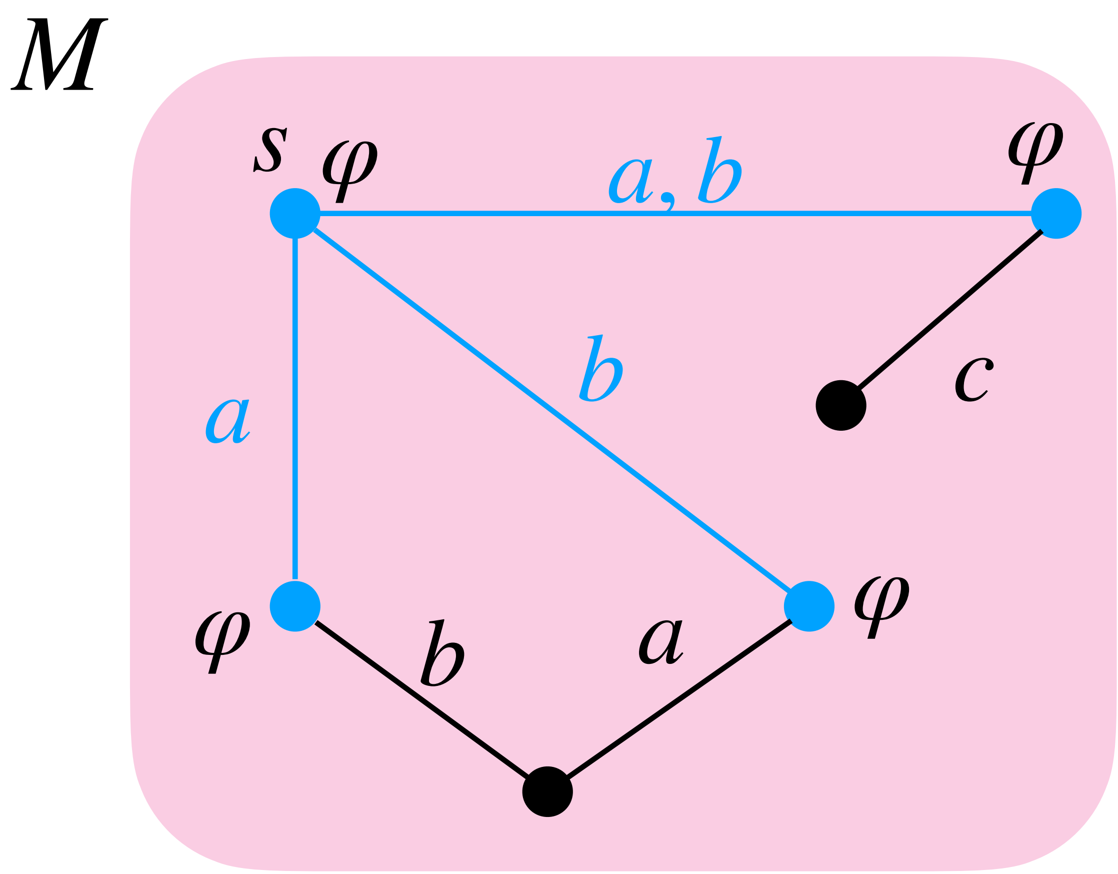

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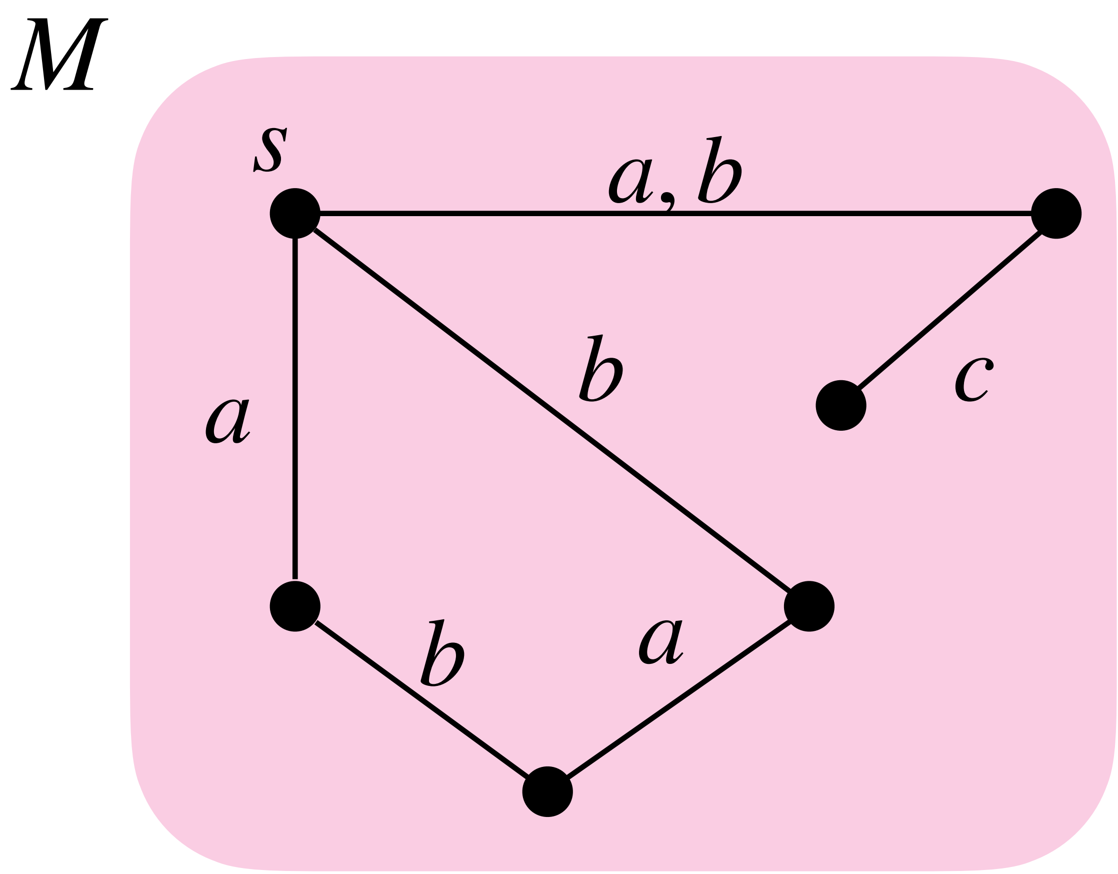

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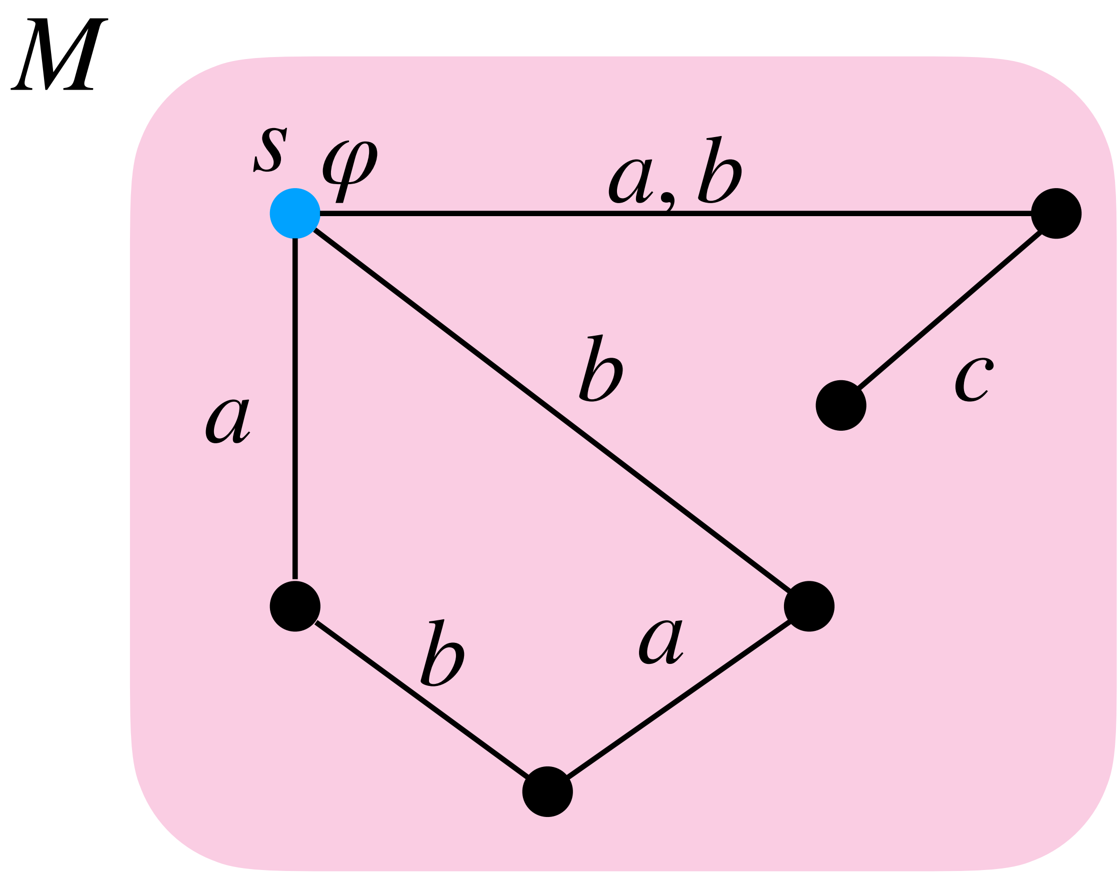

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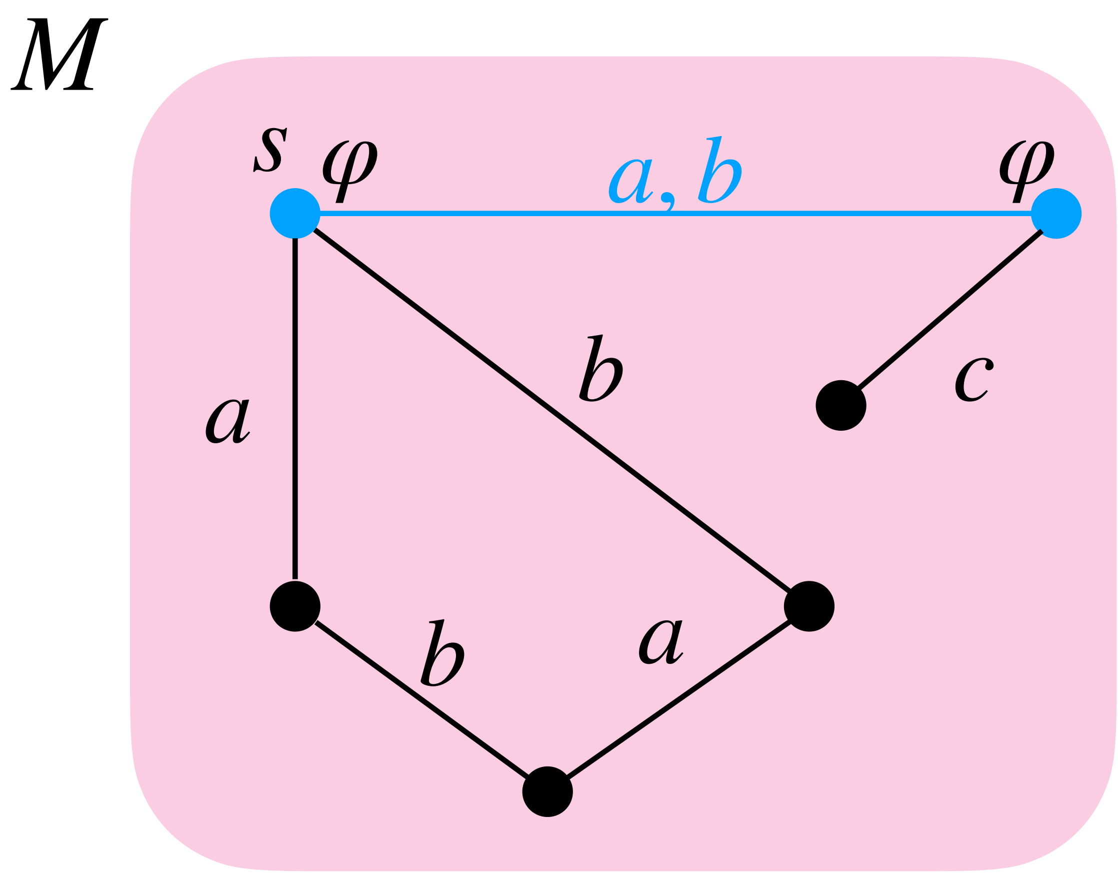

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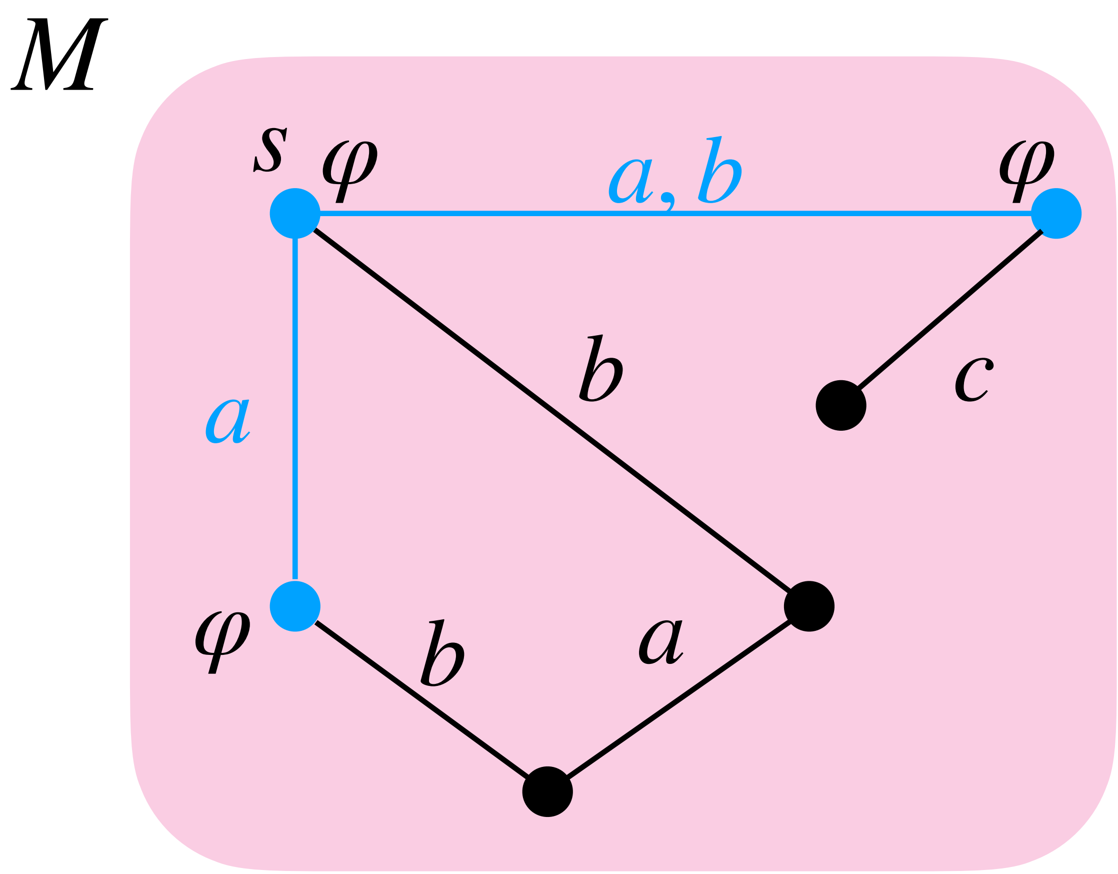

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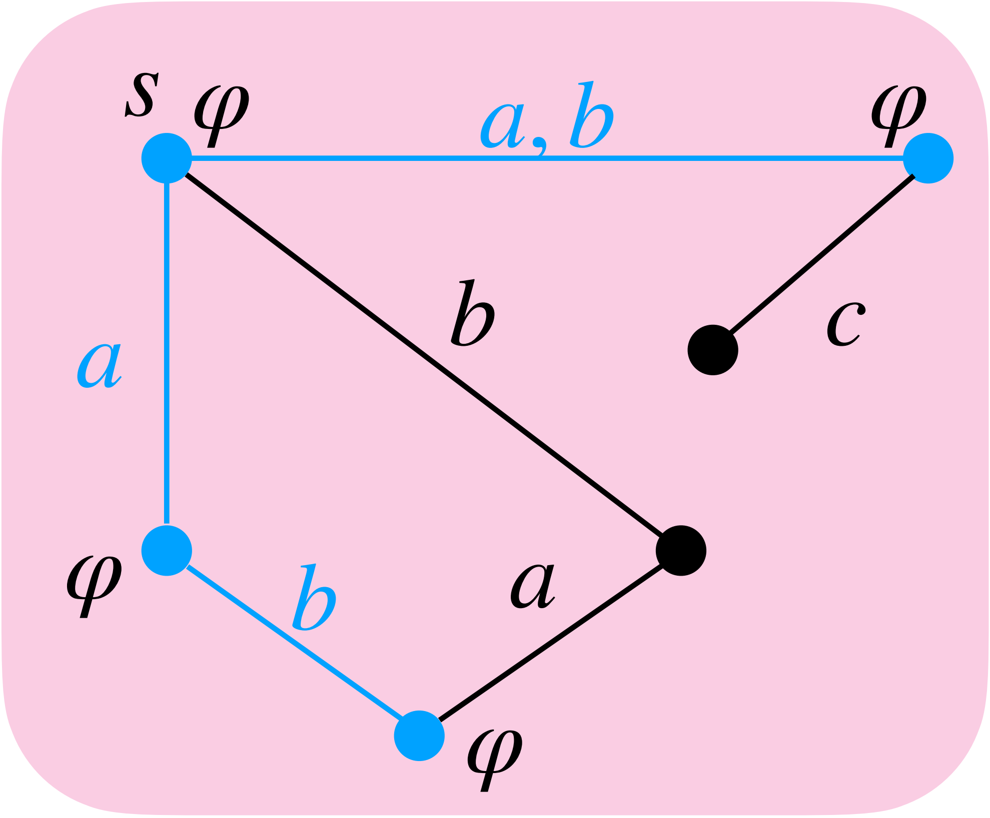

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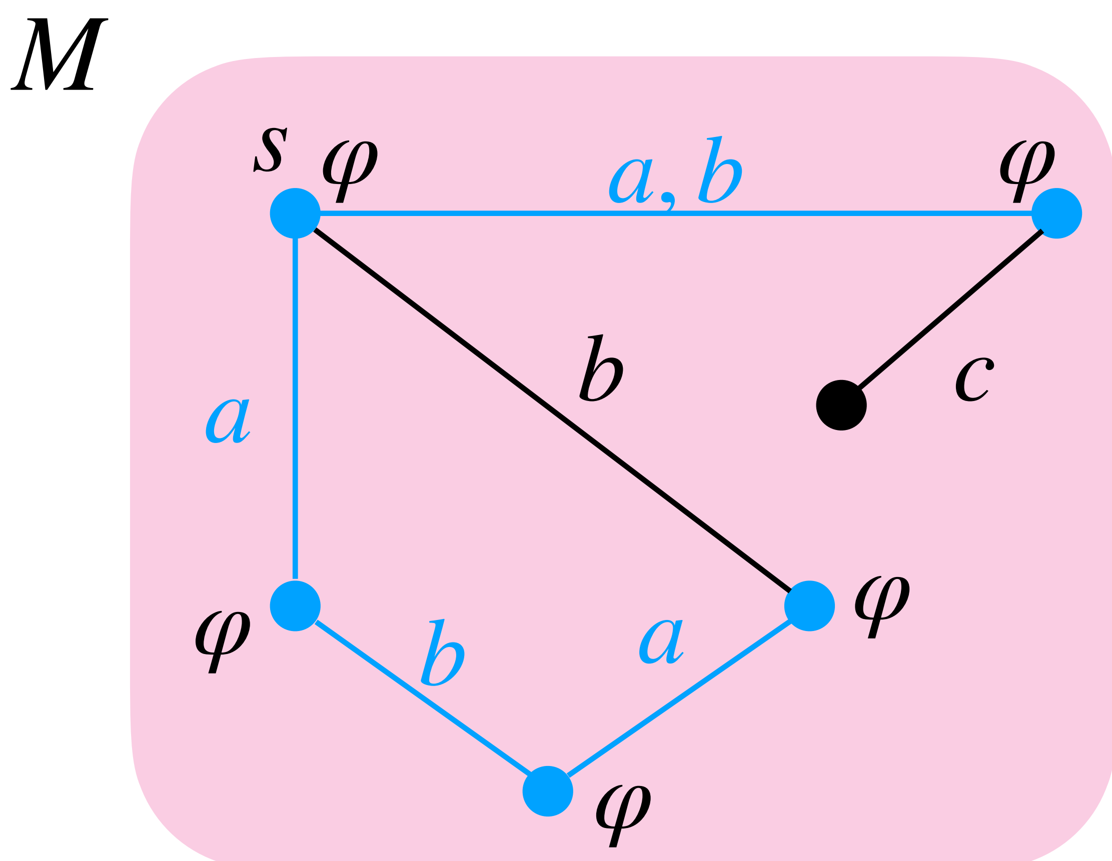

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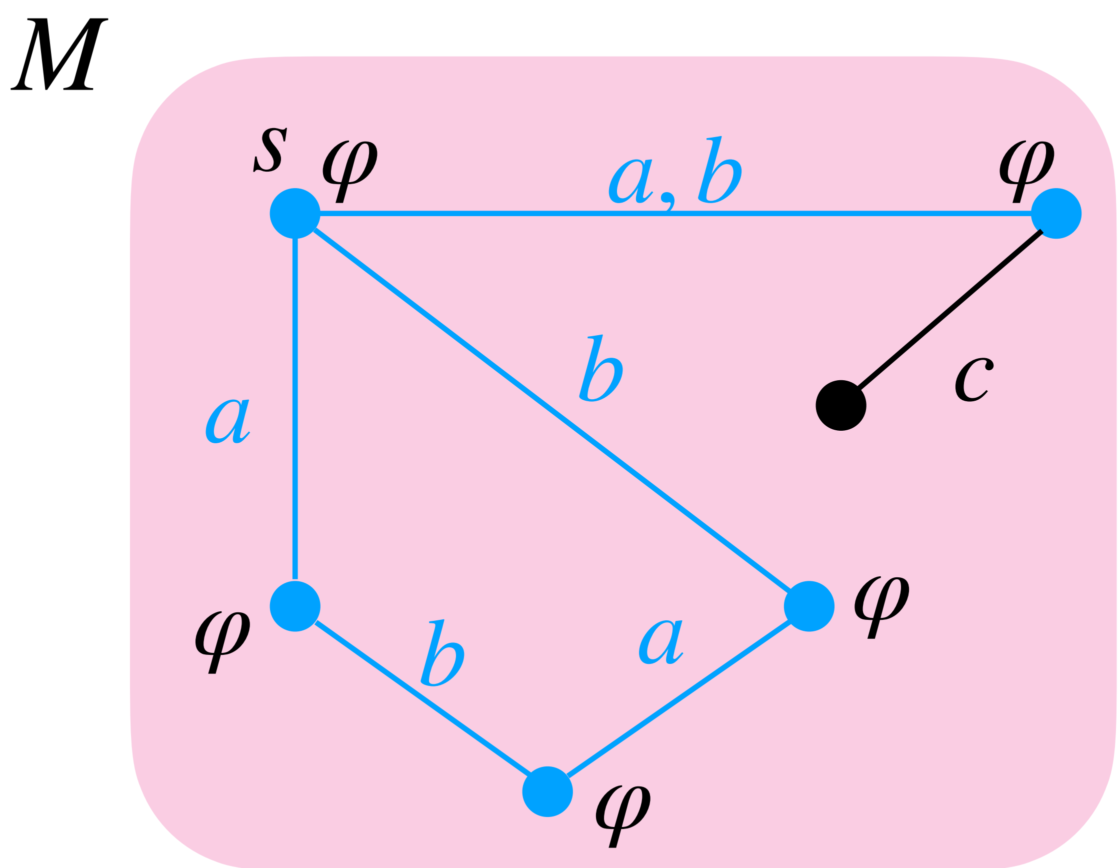

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Group Knowledge and Announcements

Why consider group knowledge?

More interesting epistemic goals.

Analysis of ability. Being able to achieve φ through communication as $\langle G \rangle \varphi$, or $\langle G \rangle \bigwedge_{a \in G} \Box_a \varphi$, or $\bigwedge_{a \in G} \Box_a \langle G \rangle \varphi$, or $D_G \langle G \rangle \varphi$, or $C_G \langle G \rangle \varphi$, and so on

Reasoning about sharing knowledge.

Sharing Knowledge

$D_G\varphi \to \langle G\rangle E_G\varphi$: Group can make its implicit knowledge explicit

$E_G\varphi \to \langle G\rangle C_G\varphi$: Group can make its knowledge common

$E_G\varphi \to \langle G\rangle C_H\varphi$: Group can share its knowledge with another group

$C_G\varphi \wedge C_H\psi \to \langle G \cup H\rangle C_{G\cup H}(\varphi \wedge \psi)$: Two groups can share their common knowledge with each other

Which of the following properties are valid?

Sharing Knowledge

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Which of the following properties are valid? None of them!

Culprit

The (in)famous offender

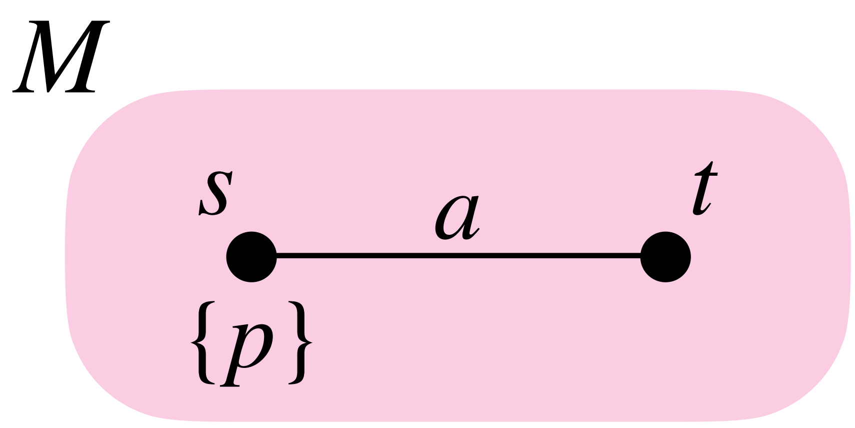
Moore sentence

$$\varphi_M := p \land \neg \Box_a p$$

Formula $[\varphi_M]\varphi_M$ is not valid on epistemic models

Take $E_G\varphi \to \langle G \rangle C_H \varphi$ with $G = \{b\}$, $H = \{a\}$, and $\varphi = \varphi_M$

$$\Box_b \varphi_M \to \langle b \rangle \Box_a \varphi_M$$



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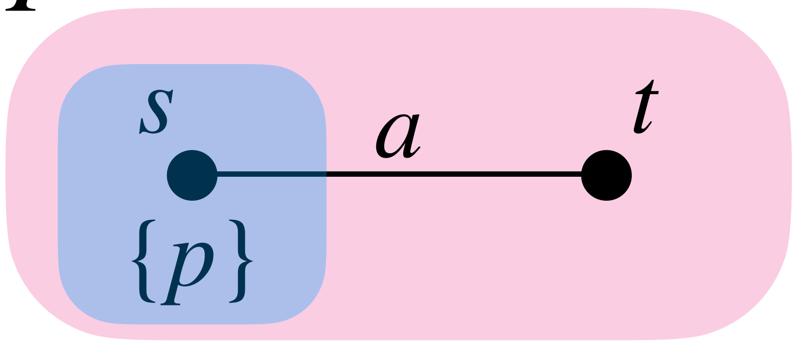

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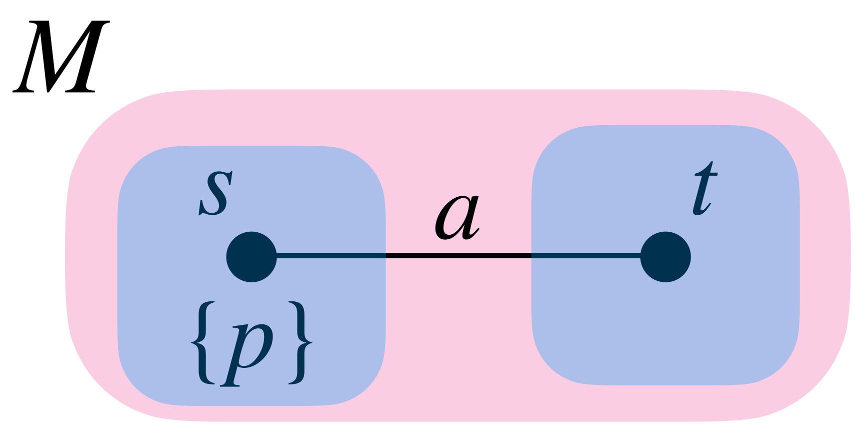

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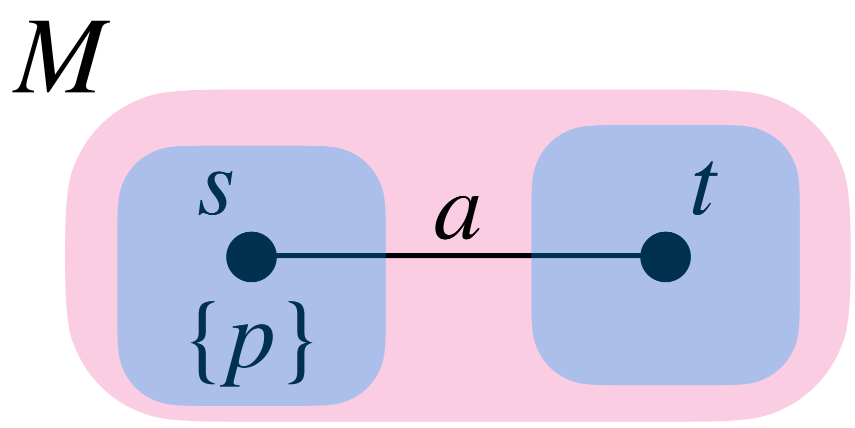

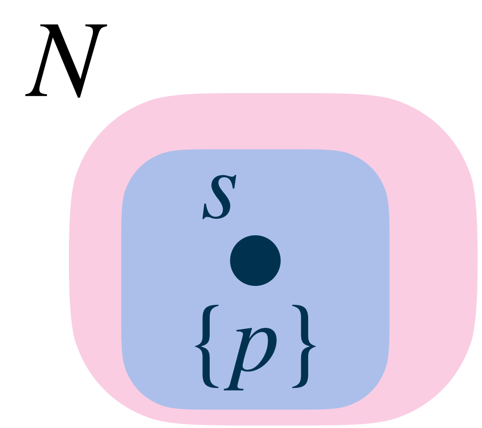

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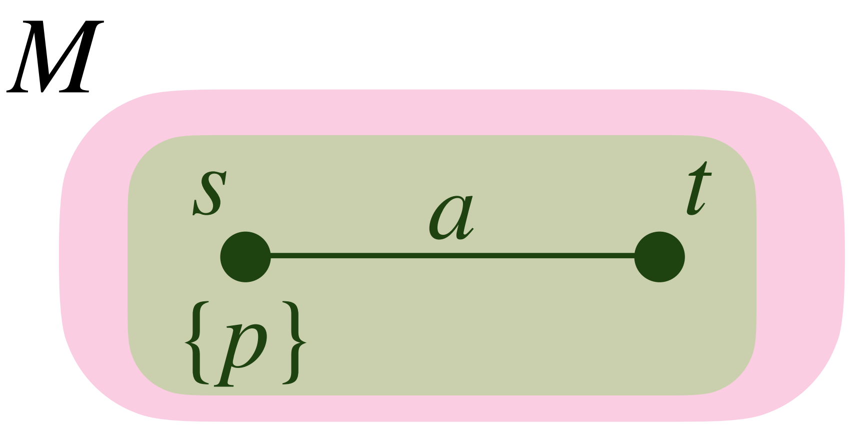

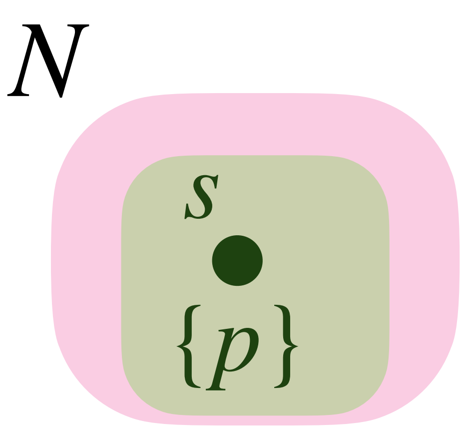

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After announcement of φ_M, agent a is no more ignorant about p

Formulas with ignorance are unstable, i.e. they tend to change their truth value after new true information was provided

So how can we reclaim some of the intuitive properties?

Get rid of instability!

Positive Fragment

$$\mathscr{EL}^+ \ni \varphi^+ ::= p \mid \neg p \mid (\varphi^+ \wedge \varphi^+) \mid (\varphi^+ \vee \varphi^+) \mid \Box_a \varphi^+$$

Theorem. $[\varphi^+]\varphi^+$ is valid

For positive formulas (stable knowledge), many of our intuitions about information sharing are valid

$$E_G\varphi^+ \to \langle G \rangle C_G \varphi^+$$

$$E_G\varphi^+ \to \langle G \rangle C_H \varphi^+$$

$$C_G\varphi^+ \wedge C_H\psi^+ \to \langle G \cup H \rangle C_{G\cup H}(\varphi^+ \wedge \psi^+)$$

$$D_G\varphi^+ \to \langle G \rangle E_G \varphi^+$$

Van Ditmarsch, Kooi. *The Secret of My Success*. 2006.