APAL with Common Knowledge

Rustam Galimullin
rustam.galimullin@uib.no
University of Bergen, Norway

Louwe B. Kuijer
lbkuijer@liverpool.ac.uk
University of Liverpool, UK

ESSLLI 2023

> LJUBLJANA > SLOVENIA

APAL with Common Knowledge

Language of APALC $\mathscr{APALC} \ni \varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box_a \varphi \mid C_G\varphi \mid [\varphi]\varphi \mid [!]\varphi$

Semantics

$M, s \models C_G\varphi$ iff $\forall n \in \mathbb{N}$: $M, s \models E_G^n\varphi$

$M, s \models [!]\varphi$ iff $\forall \psi \in \mathscr{PALC}$: $M, s \models [\psi]\varphi$

We quantify over a quantifier-free fragment

Ågotnes, RG. *Quantifying over information change with common knowledge*, 2023.

APAL with Common Knowledge

Language of APALC $\mathscr{APALC} \ni \varphi ::= p \mid \neg\varphi \mid (\varphi \land \varphi) \mid \Box_a \varphi \mid C_G\varphi \mid [\varphi]\varphi \mid [!]\varphi$

Semantics

$M, s \models C_G\varphi$ iff $\forall n \in \mathbb{N}$: $M, s \models E_G^n\varphi$

$M, s \models [!]\varphi$ iff $\forall \psi \in \mathscr{PALC} : M, s \models [\psi]\varphi$

Axioms of EL and PAL

$[!]\varphi \to [\psi]\varphi$ with $\psi \in \mathscr{PALC}$

From $\{\eta([\psi]\varphi) \mid \psi \in \mathscr{PALC}\}$ infer $\eta([!]\varphi)$

$C_G\varphi \to E_G^n\varphi$ with $n \in \mathbb{N}$

From $\{\eta(E_G^n\varphi) \mid n \in \mathbb{N}\}$ infer $\eta(C_G\varphi)$

Announcement part

Ågotnes, RG. *Quantifying over information change with common knowledge*, 2023.

APAL with Common Knowledge

Language of APALC $\mathscr{APALC} \ni \varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box_a \varphi \mid C_G\varphi \mid [\varphi]\varphi \mid [!]\varphi$

Semantics

$M, s \models C_G\varphi$ iff $\forall n \in \mathbb{N}$: $M, s \models E_G^n\varphi$

$M, s \models [!]\varphi$ iff $\forall \psi \in \mathscr{PALC} : M, s \models [\psi]\varphi$

Axioms of EL and PAL

$[!]\varphi \to [\psi]\varphi$ with $\psi \in \mathscr{PALC}$

From $\{\eta([\psi]\varphi) \mid \psi \in \mathscr{PALC}\}$ infer $\eta([!]\varphi)$

$C_G\varphi \to E_G^n\varphi$ with $n \in \mathbb{N}$

From $\{\eta(E_G^n\varphi) \mid n \in \mathbb{N}\}$ infer $\eta(C_G\varphi)$

Common knowledge part

Ågotnes, RG. *Quantifying over information change with common knowledge*, 2023.

APAL with Common Knowledge

Language of APALC $\mathscr{APALC} \ni \varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box_a \varphi \mid C_G\varphi \mid [\varphi]\varphi \mid [!]\varphi$

Semantics

$M, s \models C_G\varphi$ iff $\forall n \in \mathbb{N}$: $M, s \models E_G^n\varphi$

$M, s \models [!]\varphi$ iff $\forall \psi \in \mathscr{PALC} : M, s \models [\psi]\varphi$

Axioms of EL and PAL

$[!]\varphi \to [\psi]\varphi$ with $\psi \in \mathscr{PALC}$

$C_G\varphi \to E_G^n\varphi$ with $n \in \mathbb{N}$

From $\{\eta([\psi]\varphi) \mid \psi \in \mathscr{PALC}\}$ infer $\eta([!]\varphi)$

From $\{\eta(E_G^n\varphi) \mid n \in \mathbb{N}\}$ infer $\eta(C_G\varphi)$

Theorem. APALC is sound and complete

Ågotnes, RG. *Quantifying over information change with common knowledge*, 2023.

APAL with Common Knowledge

Language of APALC $\mathscr{APALC} \ni \varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box_a \varphi \mid C_G\varphi \mid [\varphi]\varphi \mid [!]\varphi$

Semantics

$M, s \models C_G\varphi$ iff $\forall n \in \mathbb{N}$: $M, s \models E_G^n\varphi$

$M, s \models [!]\varphi$ iff $\forall \psi \in \mathscr{PALC}$: $M, s \models [\psi]\varphi$

There is, however, a nuance that begs clarification

Recall that in (normal) APAL, we quantify over PAL, which is equivalent to EL

Does it really matter over which fragment (EL, ELC, PAL, PALC) we quantify in APALC?

Recall that even though CK can 'look' far ahead, there is always a formula with EL that can 'look' at the same distance

EL versus ELC

What do you think about EL versus ELC?

One direction. $\mathcal{EL} \subseteq \mathcal{ELC}$: ELC subsumes EL

The other direction. $\mathcal{ELC} \subseteq \mathcal{ELL}$?

Consider formula $C_{\{a,b\}} \neg p$

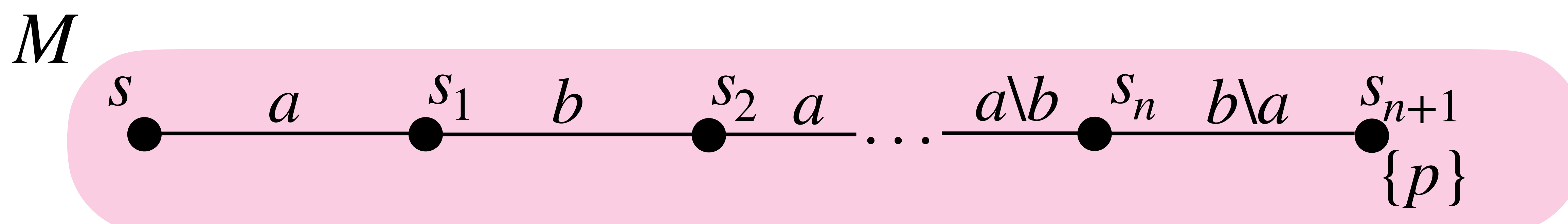
EL versus ELC

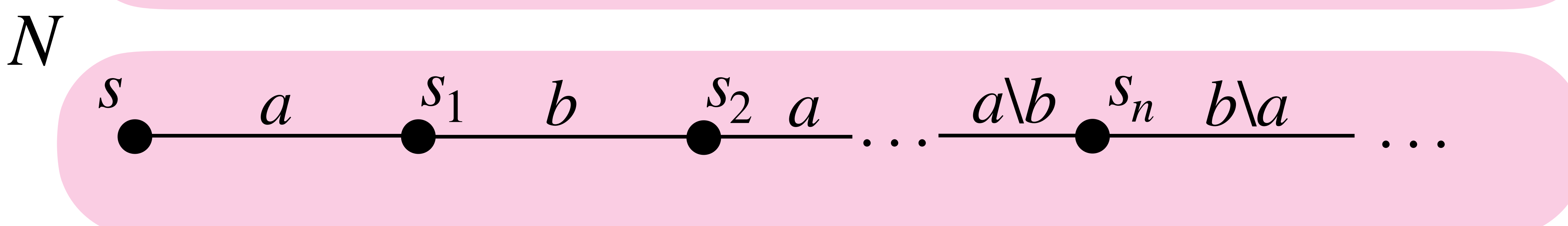
The other direction. $\mathcal{ELC} \subseteq \mathcal{EL}$?

Consider formula $C_{\{a,b\}}\neg p$

Assume that there is an equivalent $\psi \in \mathcal{EL}$

Since ψ is finite, it has some finite modal depth n





In which model is $C_{\{a,b\}}\neg p$ true?

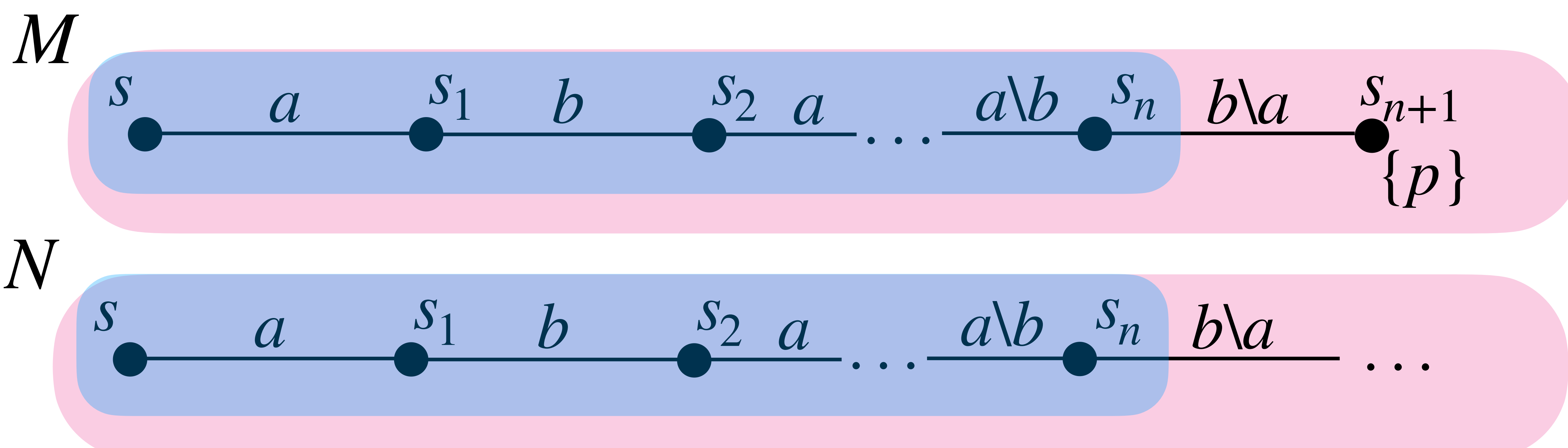
EL versus ELC

The other direction. $\mathscr{ELC} \subseteq \mathscr{EL}$?

Consider formula $C_{\{a,b\}} \neg p$

Assume that there is an equivalent $\psi \in \mathscr{EL}$

Since ψ is finite, it has some finite modal depth n



Cannot find the difference with an EL formula!

EL versus ELC

Theorem. ELC is strictly more expressive than EL

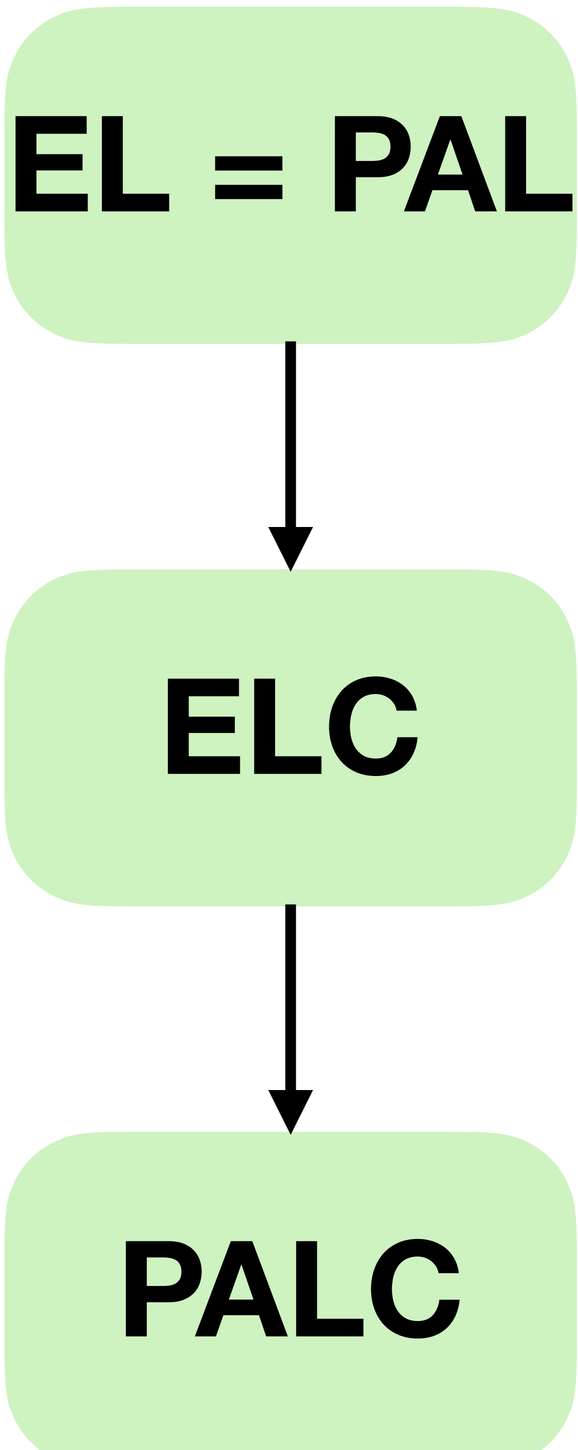
Corollary. ELC is strictly more expressive than PAL

What about PALC? Do we gain anything compared to ELC?

Theorem. PALC is strictly more expressive than ELC

Proof intuition. Public announcements can remove states ‘far away’, and this difference can be reached by CK and not always by standard knowledge (finite modal depth)

The EL Landscape



Sooooo….

Which fragment we quantify over in APALC may matter

On the one hand, expressivity of EL, ELC, and PALC is different

On the other hand, maybe quantifying over formulas of arbitrary modal depth negates the power of common knowledge

The EL Landscape

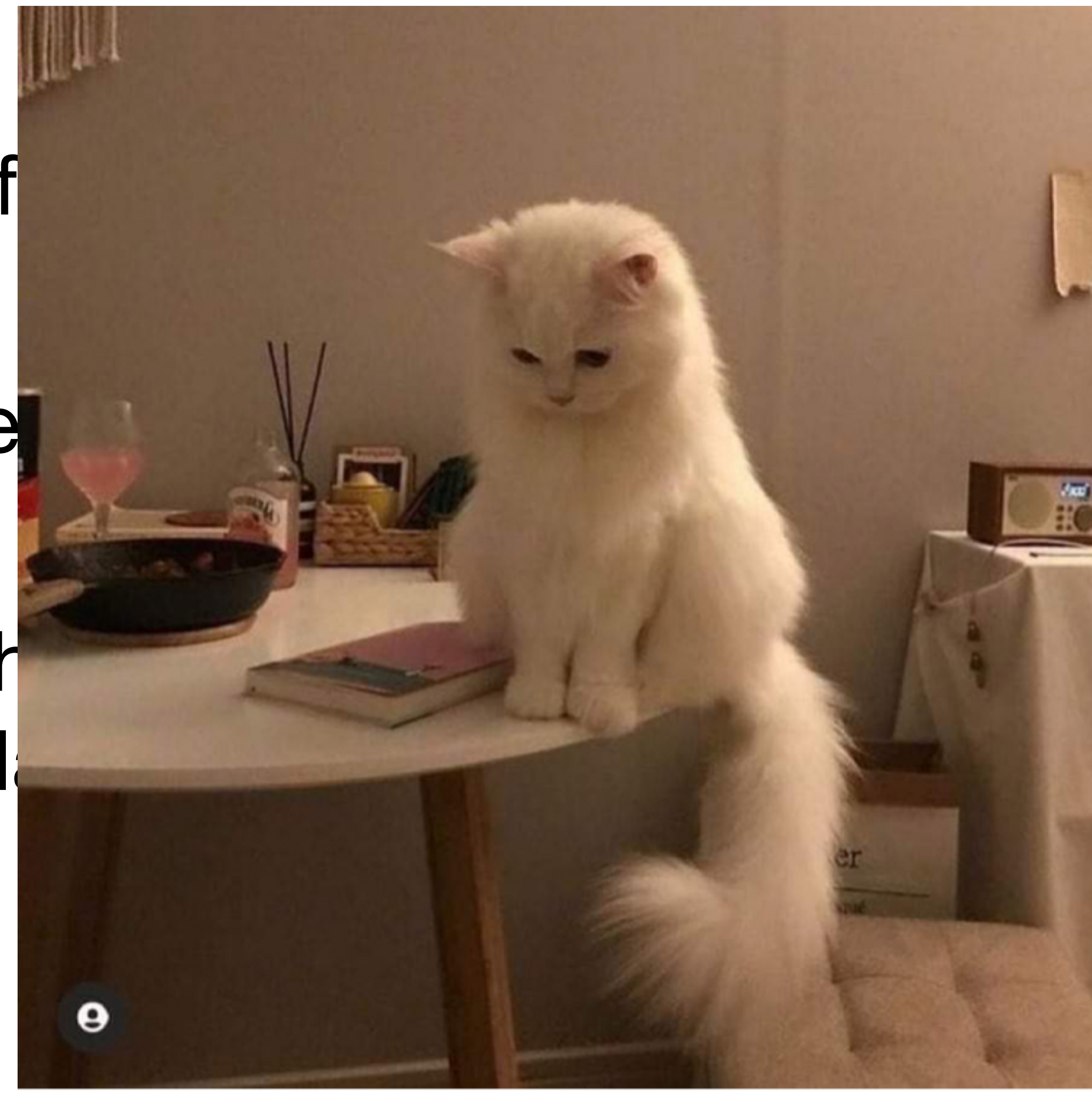
EL = PAL

↓

ELC

↓

PALC

Which f  APALC may

On the ... , ELC, and

On th ... ving over

formul ... egates the

ge

APALs with Common Knowledge

APALC = PALC + $[!]\varphi$

$$M, s \models [!]\varphi \text{ iff } \forall \psi \in \mathcal{PAL} : M, s \models [\psi]\varphi$$

APALC^X **= PALC +** $[!]^X\varphi$

$$M, s \models [!]^X\varphi \text{ iff } \forall \psi \in \mathcal{ELC} : M, s \models [\psi]\varphi$$

APALC^{XX} **= PALC +** $[!]^{XX}\varphi$

$$M, s \models [!]^{XX}\varphi \text{ iff } \forall \psi \in \mathcal{PALC} : M, s \models [\psi]\varphi$$

Ågotnes, RG. *Quantifying over information change with common knowledge*, 2023.

APALC versus APALC^X

APALC = PALC + $[!]\varphi$

$M, s \models [!]\varphi$ iff $\forall \psi \in \mathscr{PAL} : M, s \models [\psi]\varphi$

APALC^X = PALC + $[!]^X\varphi$

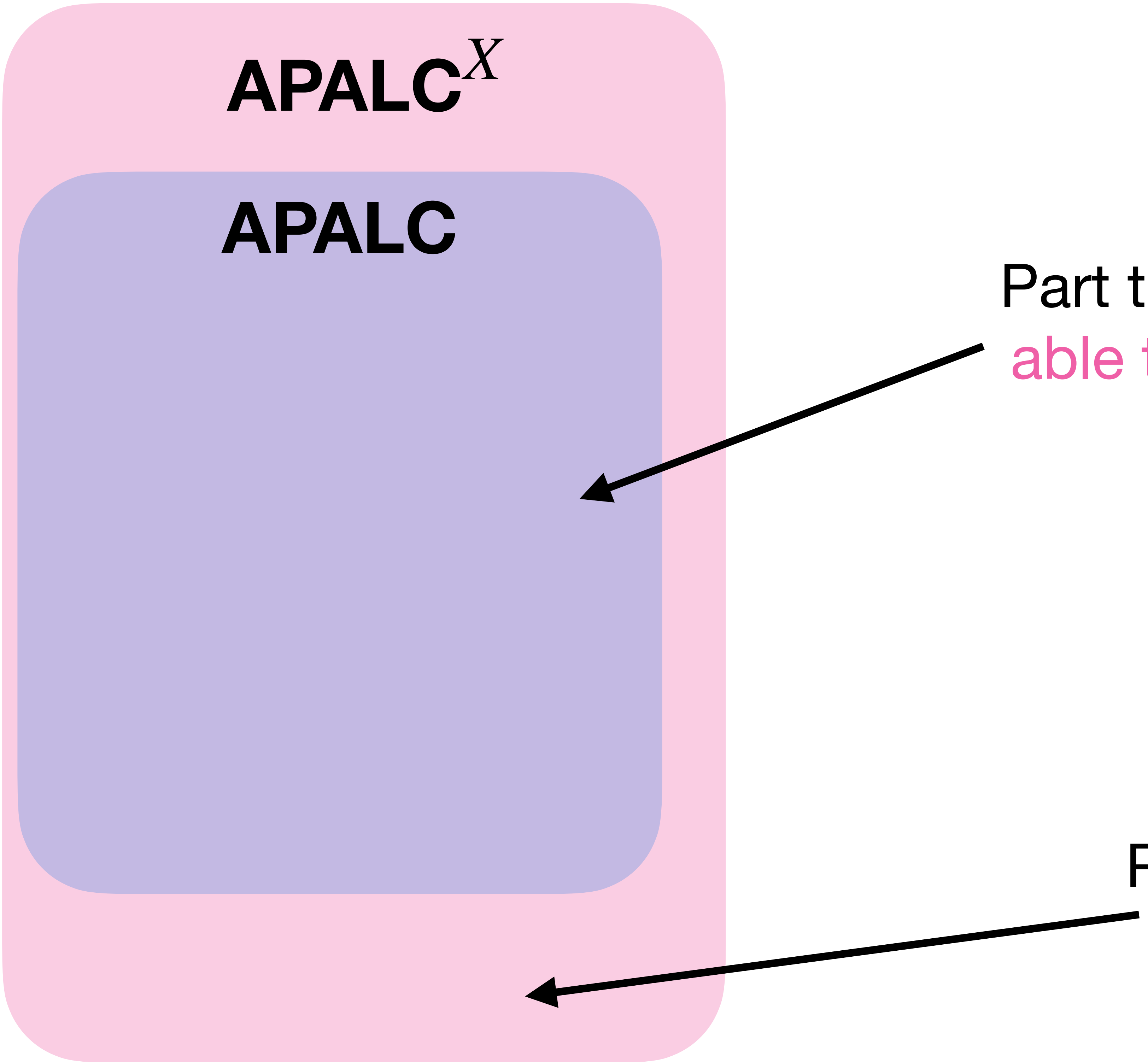
$M, s \models [!]^X\varphi$ iff $\forall \psi \in \mathscr{ELC} : M, s \models [\psi]\varphi$

Are there two (classes of) models that APALC^X can distinguish but APALC cannot?

What is the difference in the models that we are looking for?

What should be the same in the models?

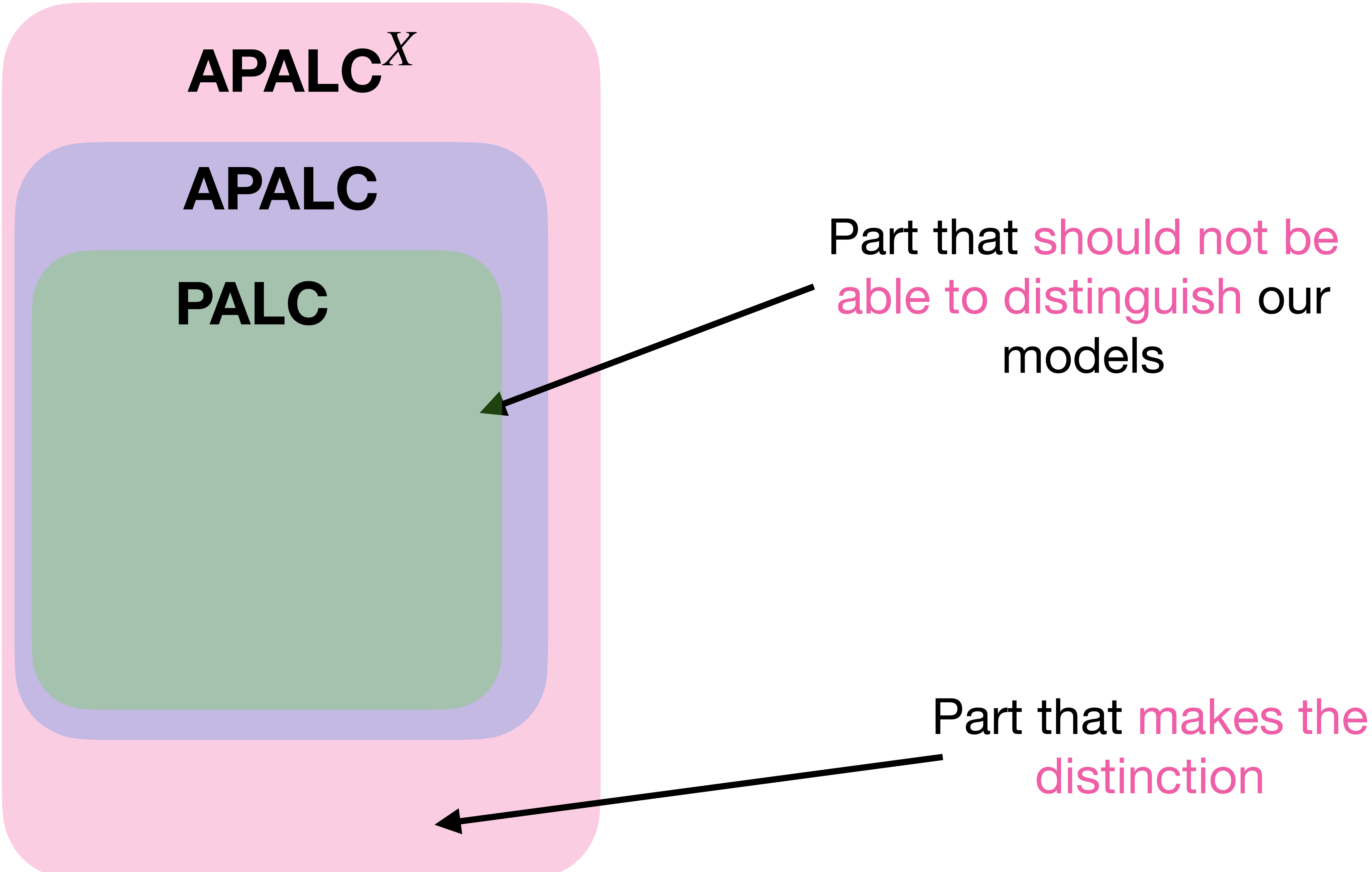
How I Think



Part that should not be able to distinguish our models

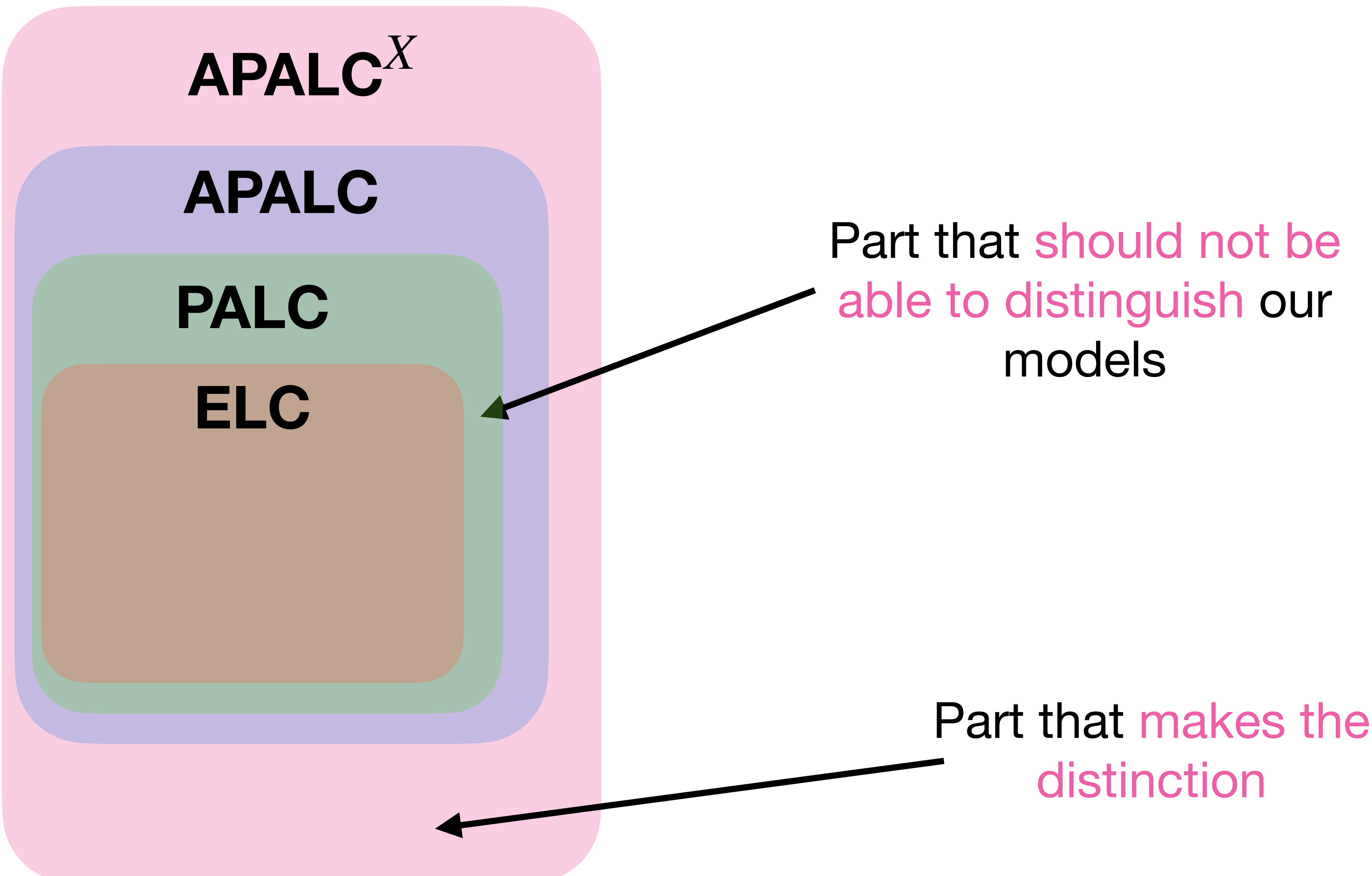
Part that makes the distinction

How I Think



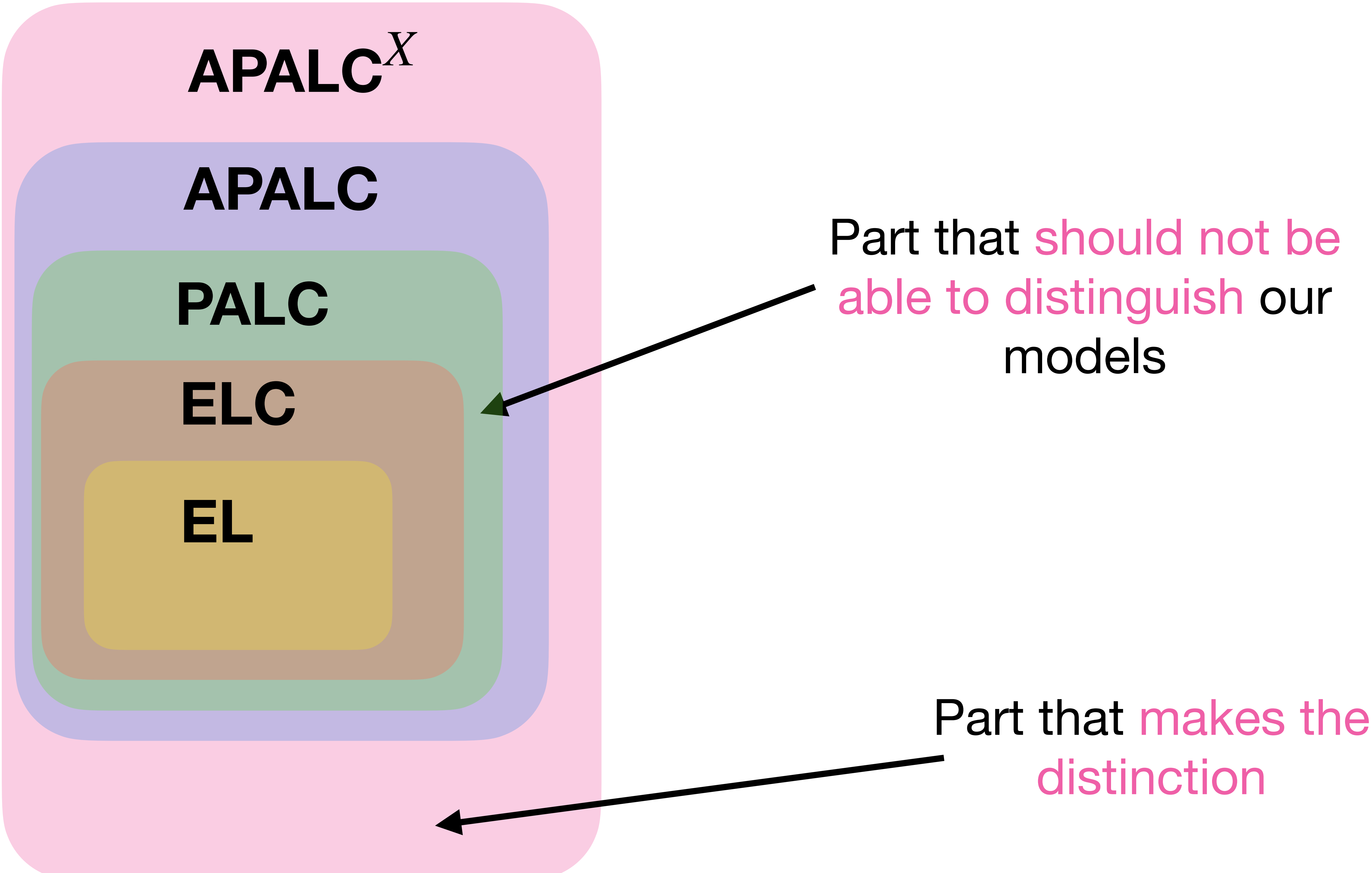
APALC^X
APALC
PALC
Part that should not be able to distinguish our models
Part that makes the distinction

How I Think

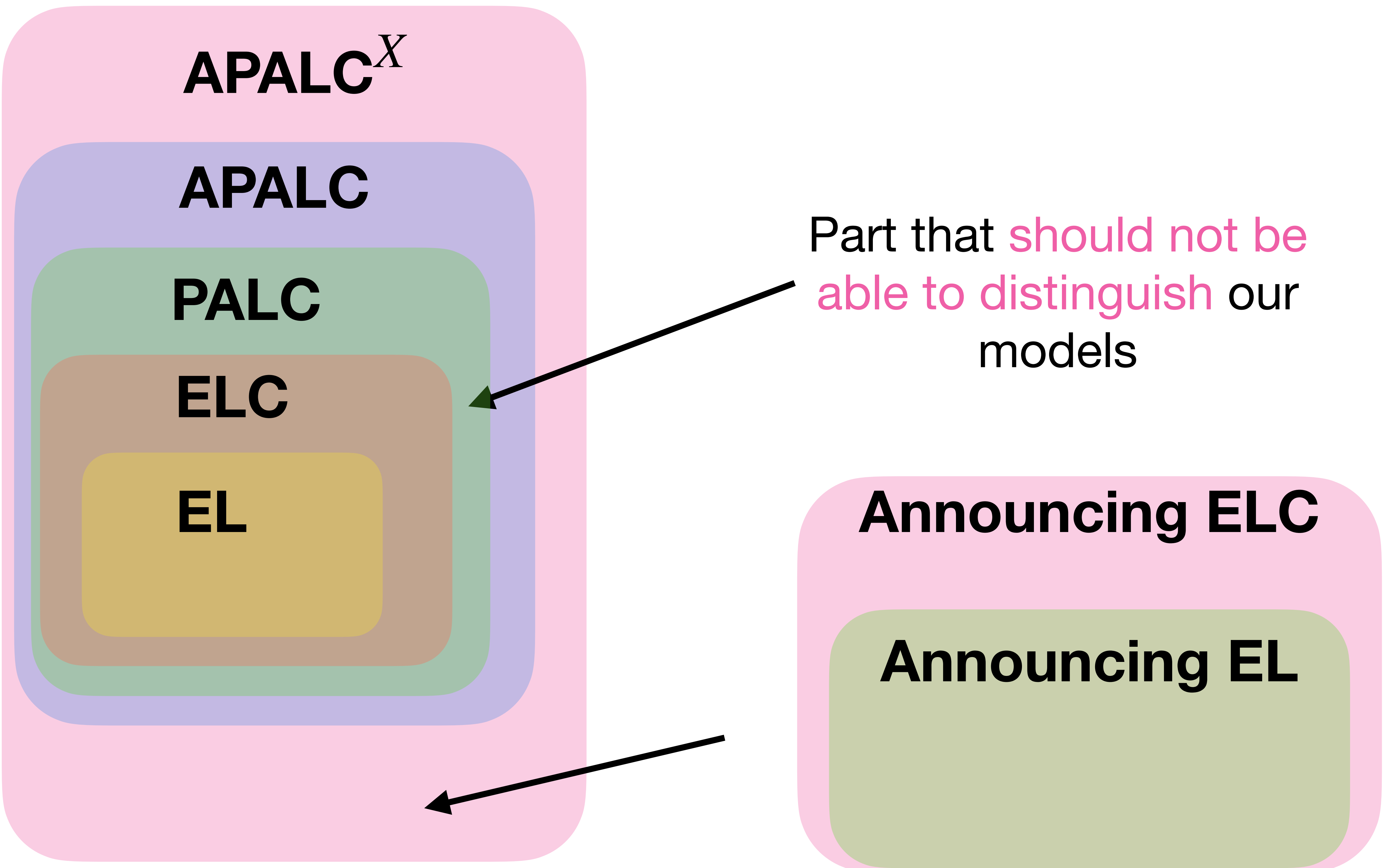


$APALC^X$
APALC
PALC
ELC
Part that should not be able to distinguish our models
Part that makes the distinction

How I Think

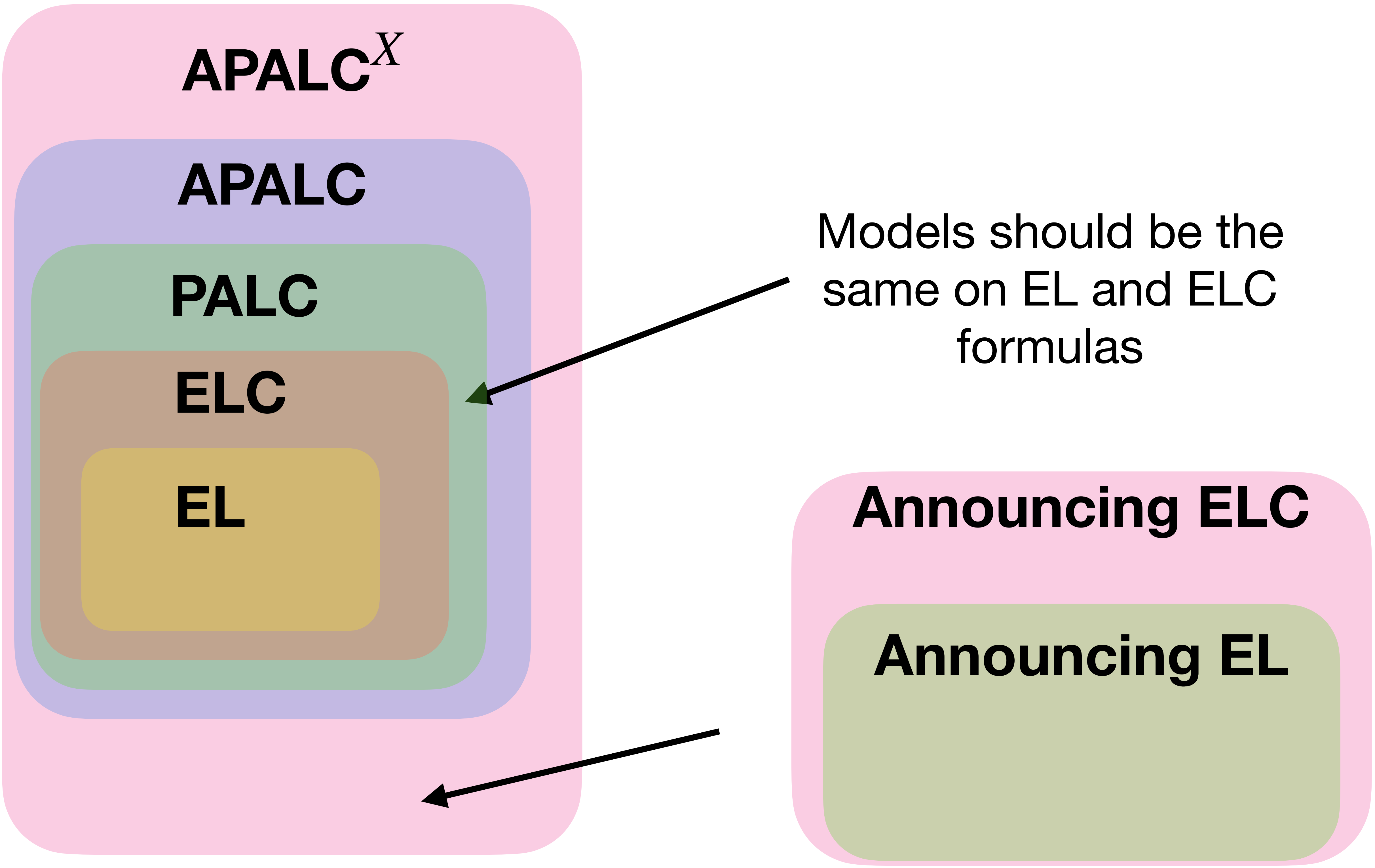

$APALC^X$
APALC
PALC
ELC
EL
Part that should not be able to distinguish our models
Part that makes the distinction

How I Think



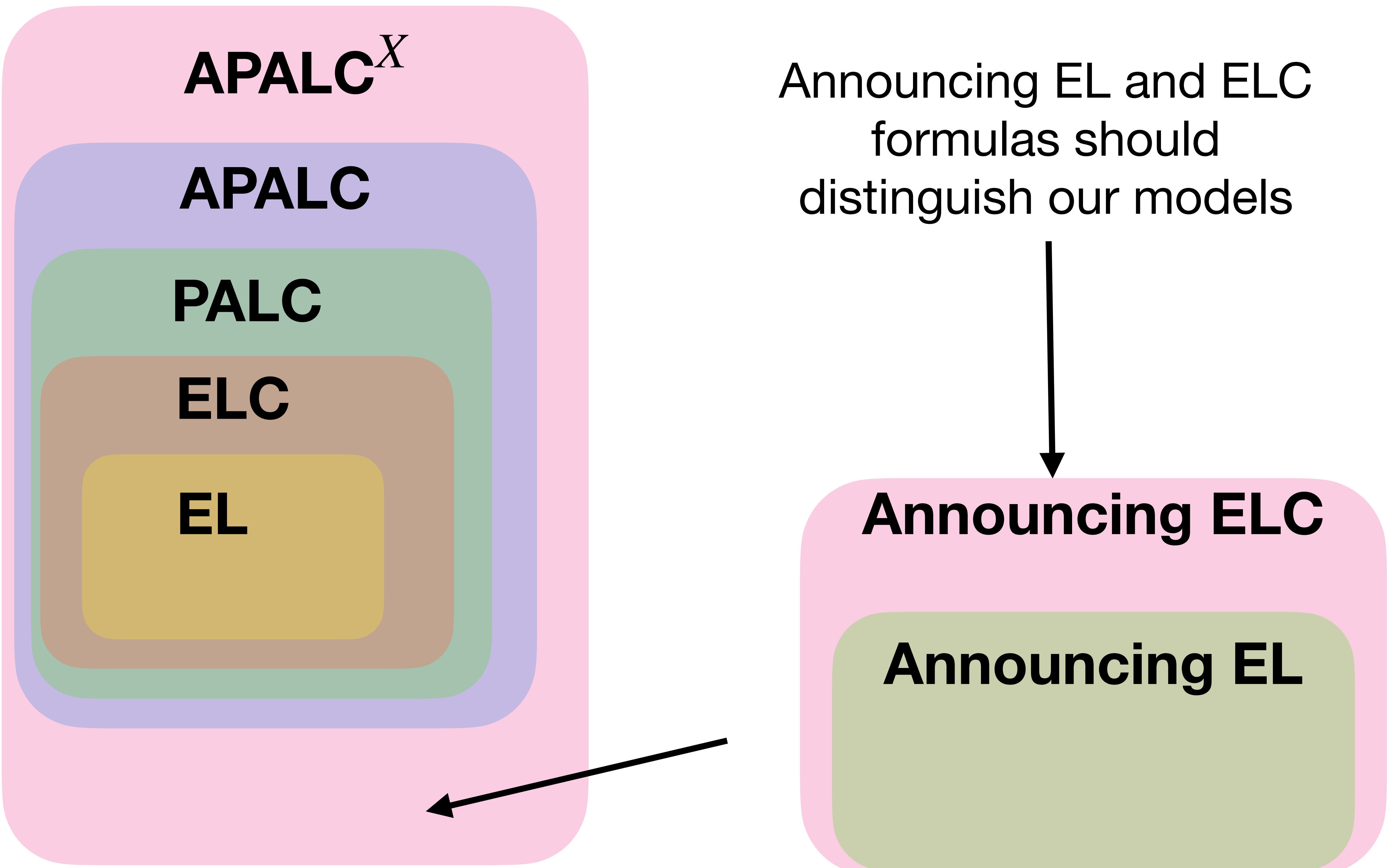
$APALC^X$
APALC
PALC
ELC
EL
Part that should not be able to distinguish our models
Announcing ELC
Announcing EL

How I Think

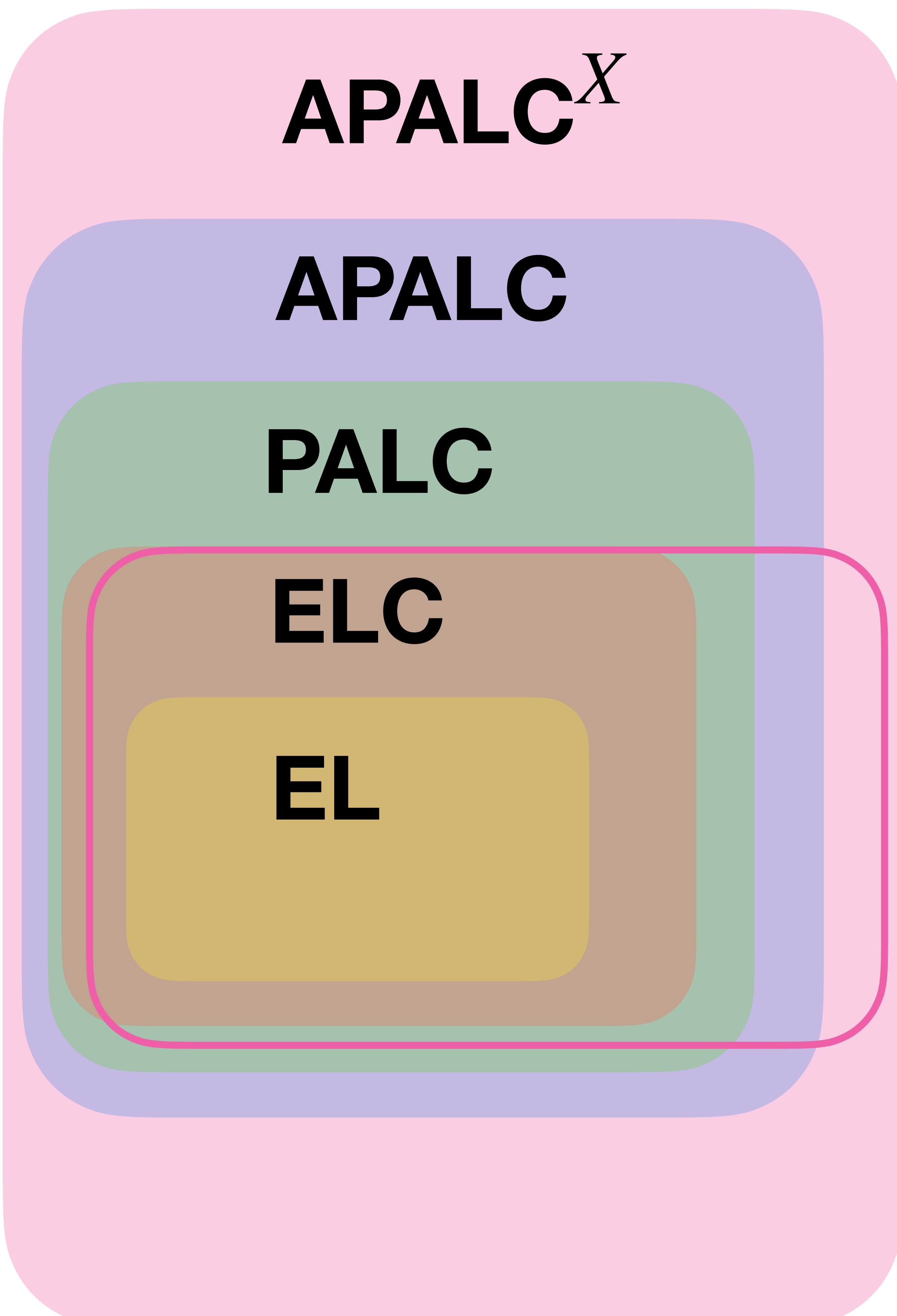


APALC^X
APALC
PALC
ELC
EL
Models should be the same on EL and ELC formulas
Announcing ELC
Announcing EL

How I Think


$\mathbf{APALC}^X$
APALC
PALC
ELC
EL
Announcing EL and ELC formulas should distinguish our models
Announcing ELC
Announcing EL

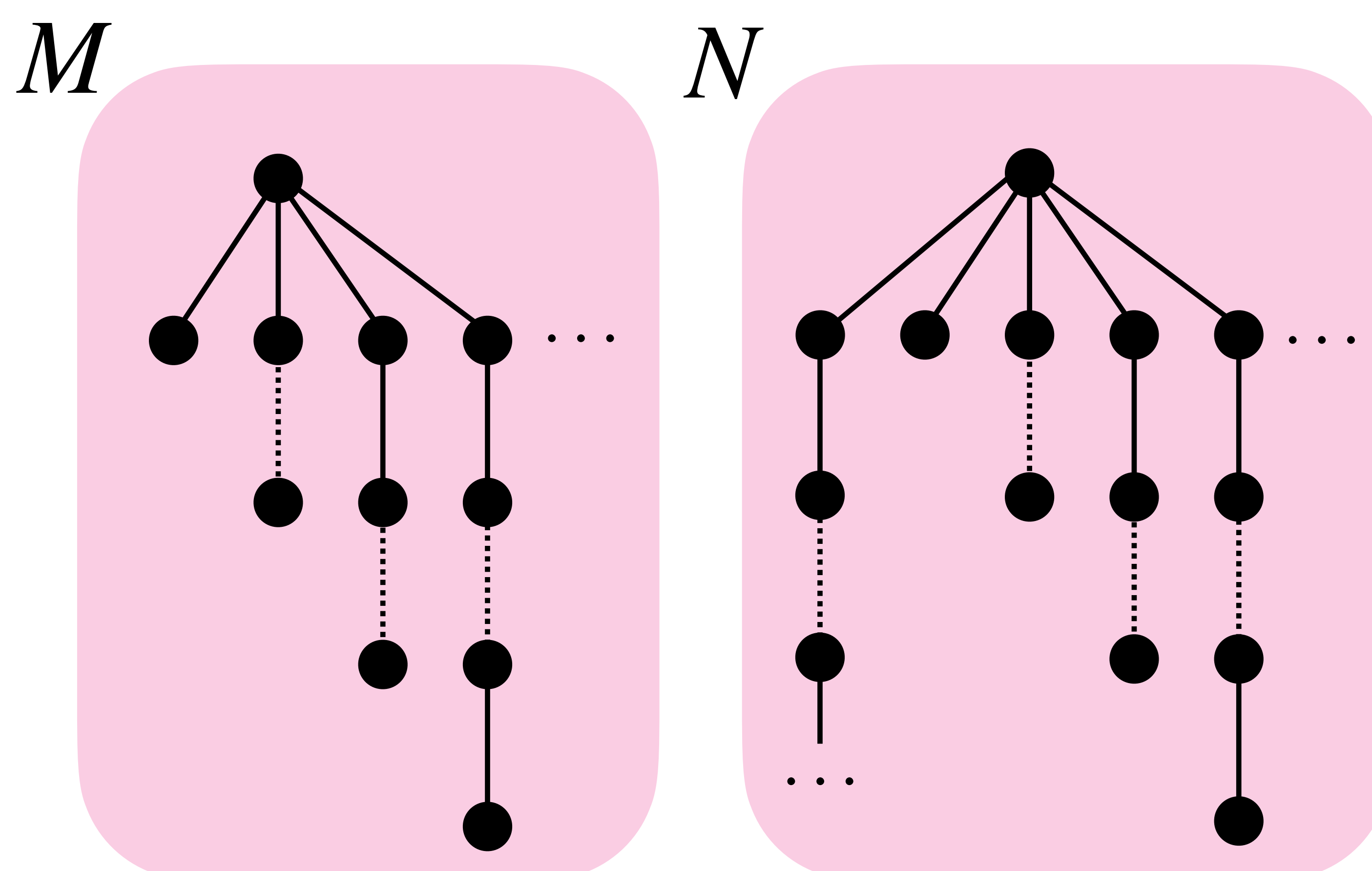
How I Think



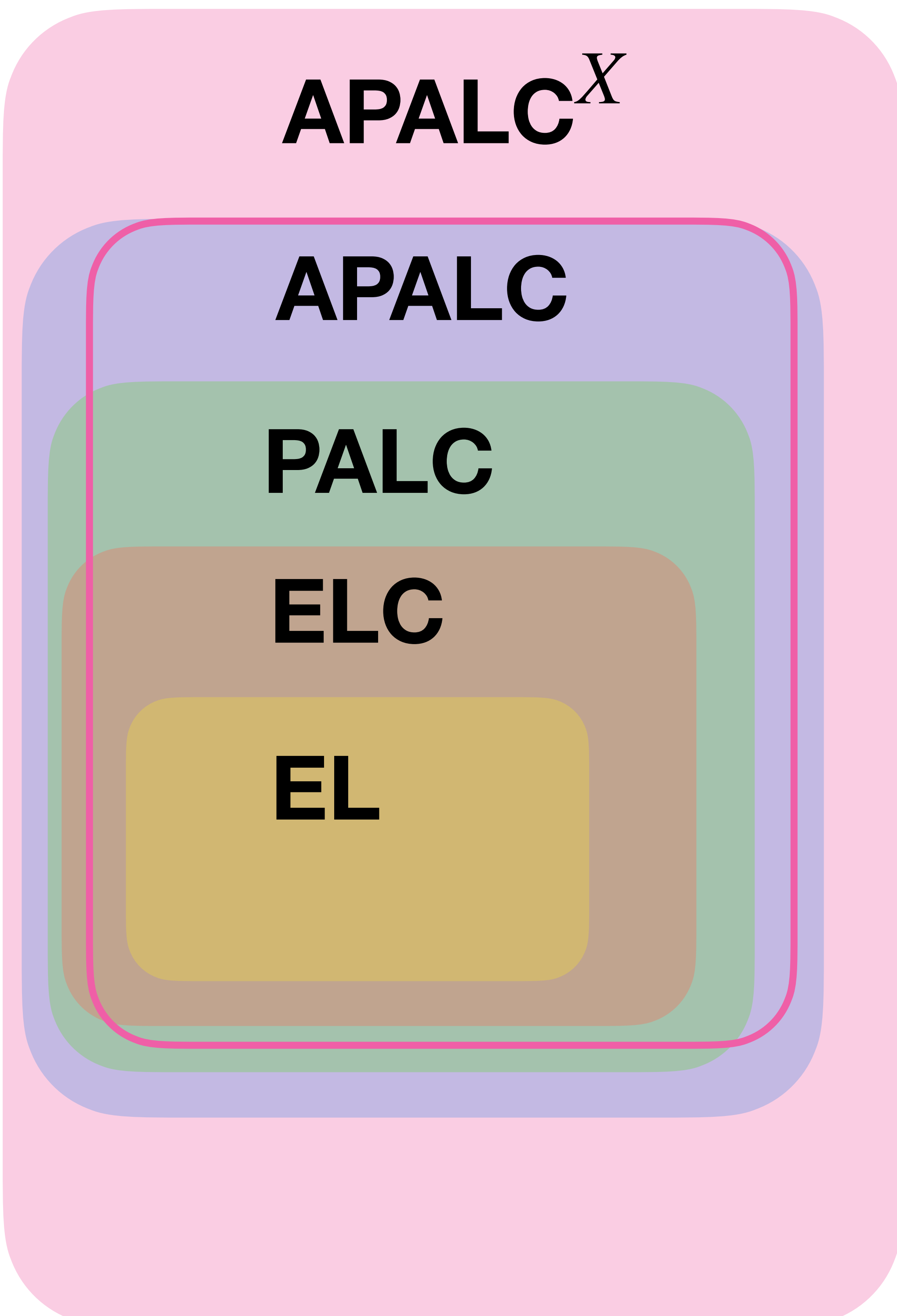
The sameness part

We want EL-non-distinguishable models that are not bisimilar

What happens with EL announcements on the models?



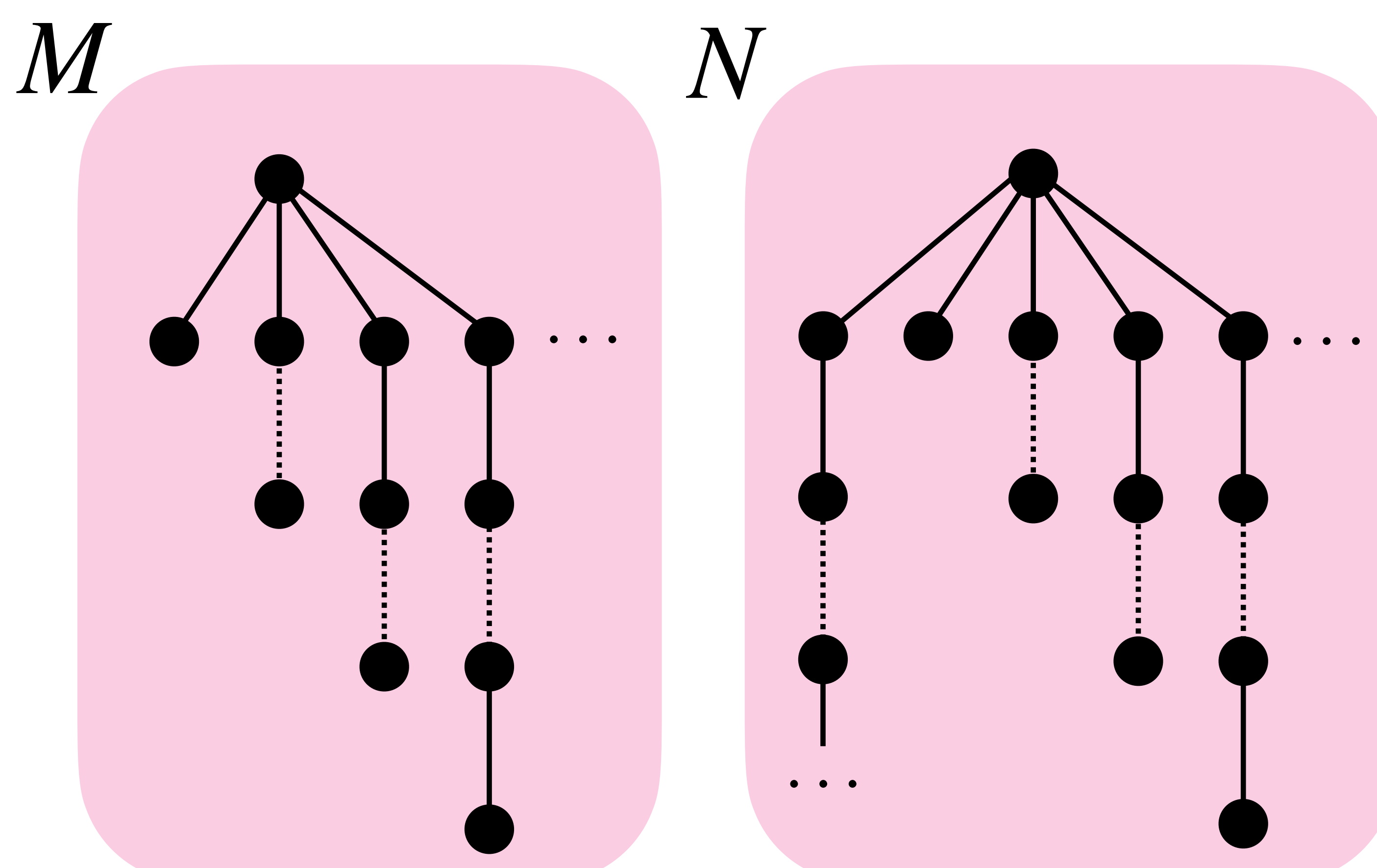
How I Think



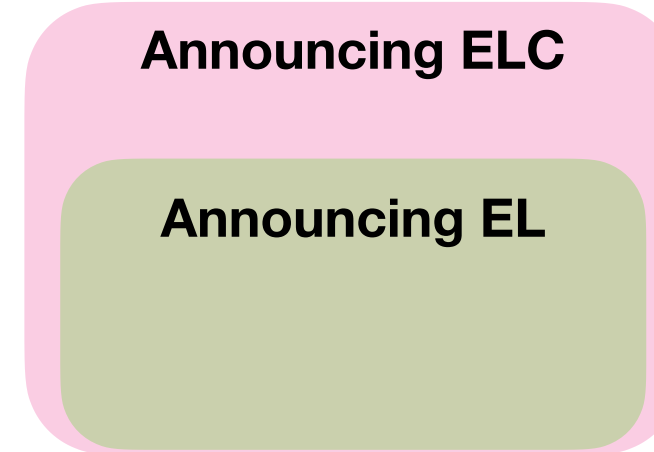
The sameness part

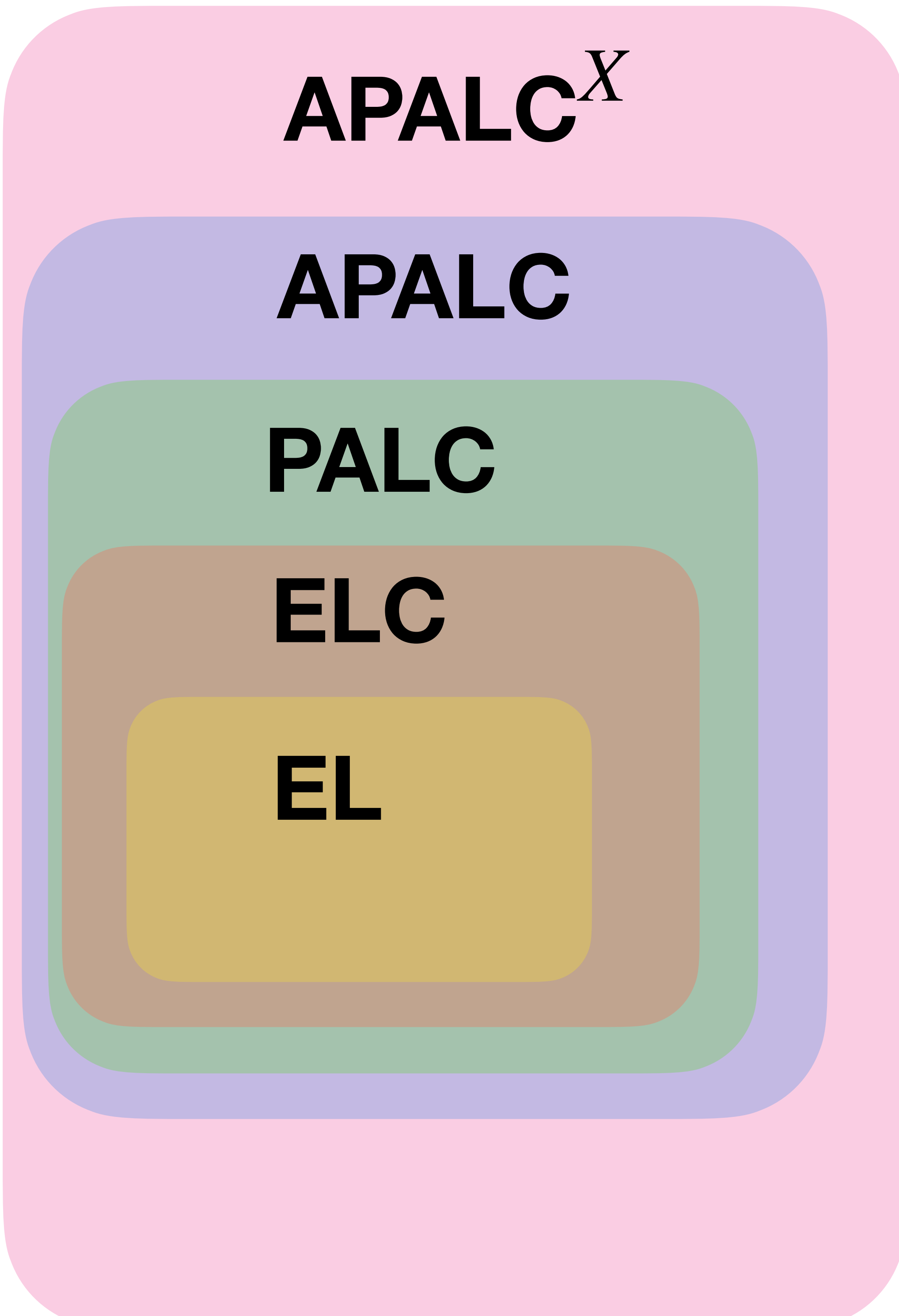
We want EL-non-distinguishable models that are not bisimilar

What happens with EL announcements on the models?



How I Think

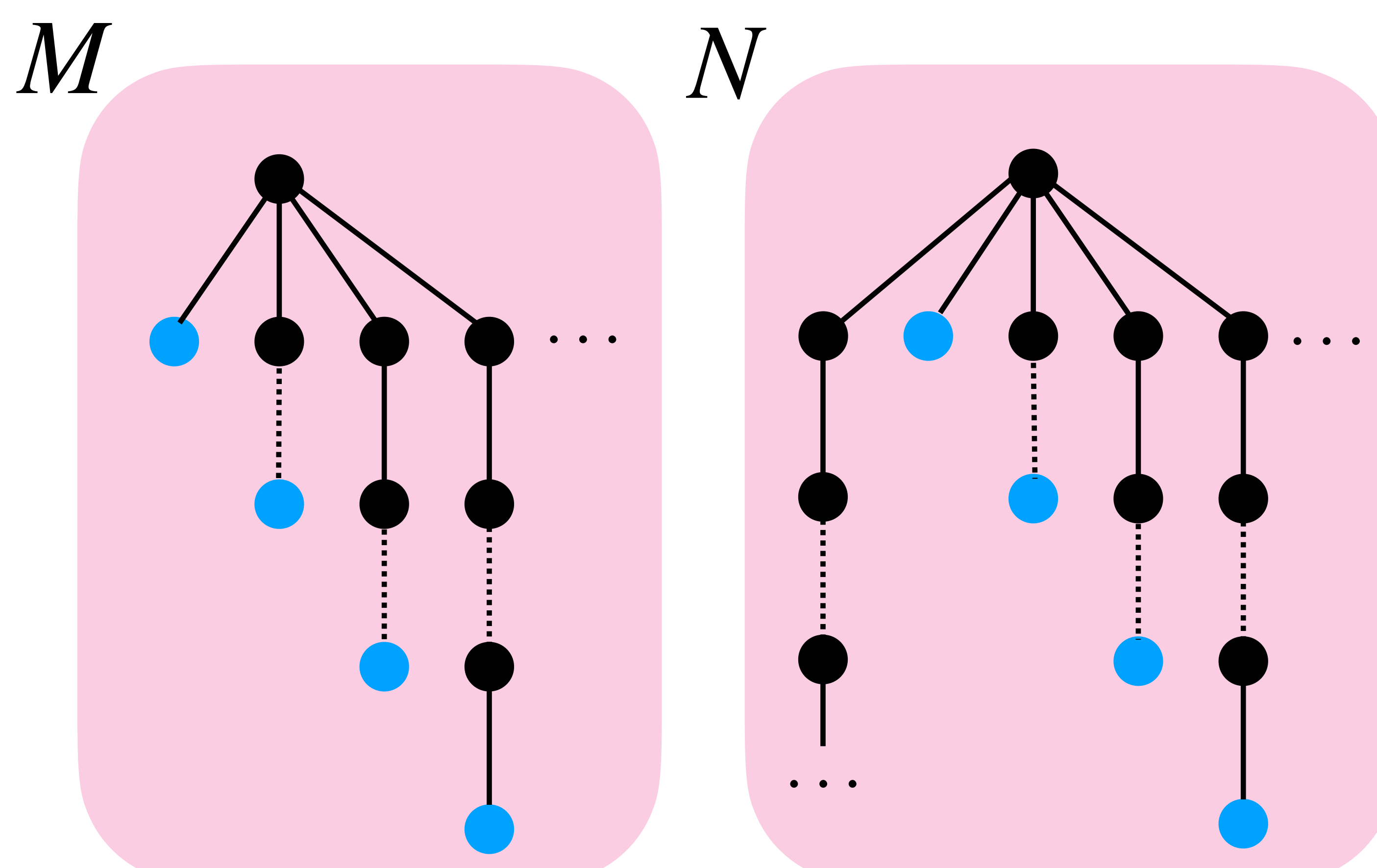




The difference part

We can assume that q that does not appear explicitly

q helps us to distinguish finite and the infinite chains



APALC versus APALC^X

We can combine all these intuitions (and a little bit more) to provide a bisimulaiton-based argument

Theorem. The are (classes of) models that APALC^X can distinguish and APALC cannot

The other direction is even more interesting: does greater scope of quantification in APALC^X translate into greater expressivity?

Theorem. The are (classes of) models that APALC can distinguish and APALC^X cannot

Quantifier $[!]^X$ sometimes is too powerful to notice a difference

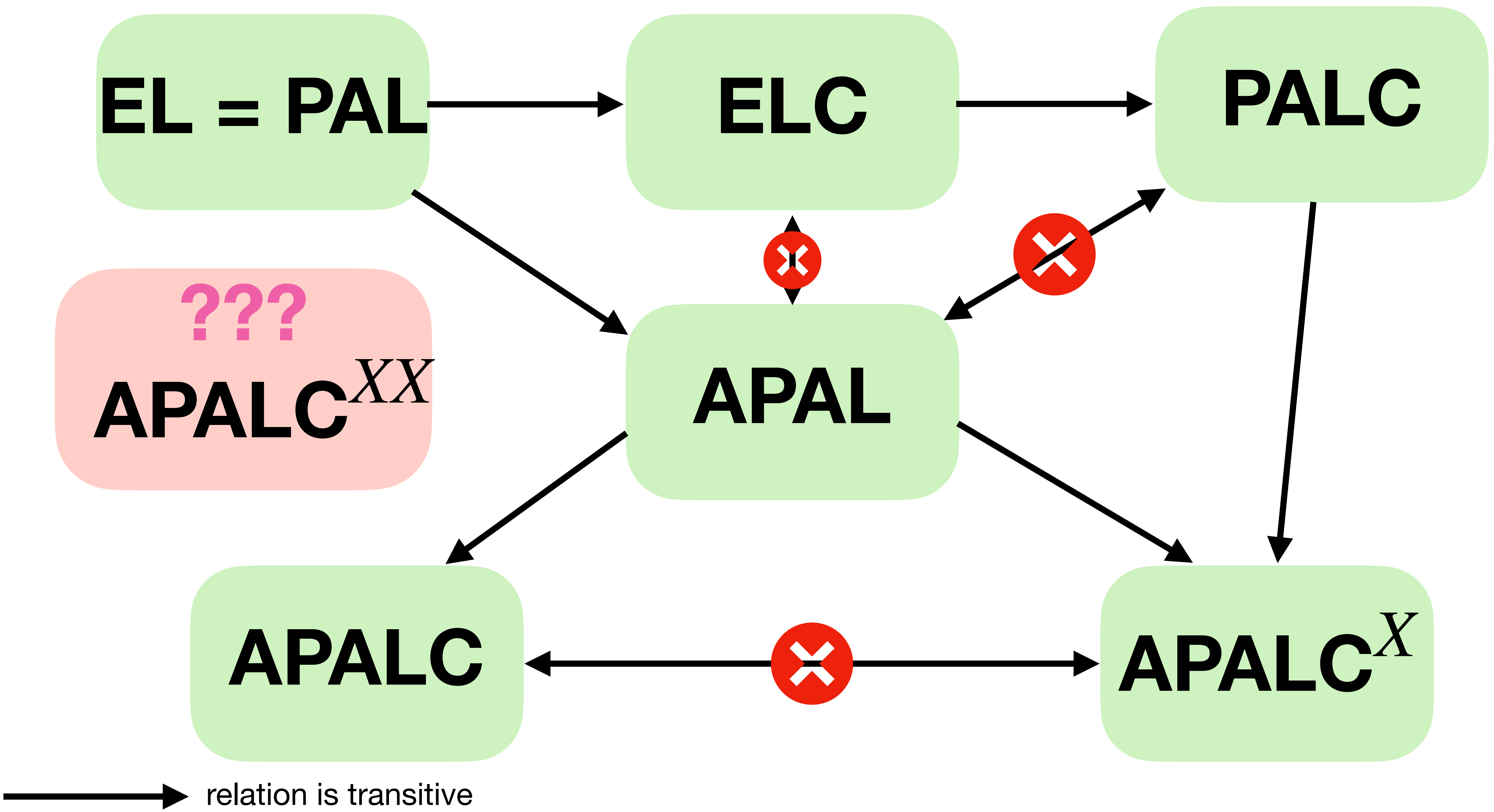
X versus XX

What about $APALC^{XX}$?

X versus XX

What about APALC^{XX}? I don't know!

APALC landscape



Overview of APALC

Axioms of EL and PAL

$[!]\varphi \to [\psi]\varphi$ with $\psi \in \mathscr{L}$

From $\{\eta([\psi]\varphi) \mid \psi \in \mathscr{L}\}$ infer $\eta([!]\varphi)$

$C_G\varphi \to E_G^n\varphi$ with $n \in \mathbb{N}$

From $\{\eta(E_G^n\varphi) \mid n \in \mathbb{N}\}$ infer $\eta(C_G\varphi)$

Variants.

APALC: $\mathscr{PAL}$, $[!]\varphi$

APALC^X: $\mathscr{ELC}$, $[!]^X\varphi$

APALC^{XX}: $\mathscr{PALC}$, $[!]^{XX}\varphi$

Theorem. APALCs are more expressive than APAL

Theorem. APALC and APALC^X are incomparable

Open Problem. Expressivity of APALC^{XX}

Ågotnes, RG. *Quantifying over information change with common knowledge*, 2023.

Alternative Open Problem

Open Problem*. Is there a finitary axiomatisation of APAL with common knowledge?

Recurring Tiling Problem

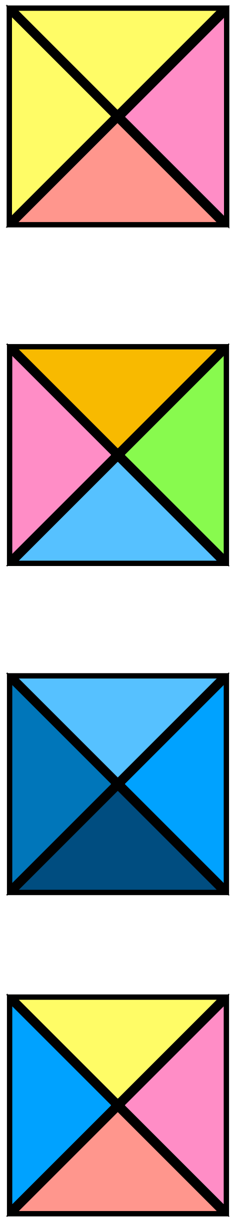
Given a finite set of colours C, a tile is a function

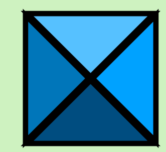
$$\tau : \{\texttt{north}, \texttt{south}, \texttt{east}, \texttt{west}\} \to C$$

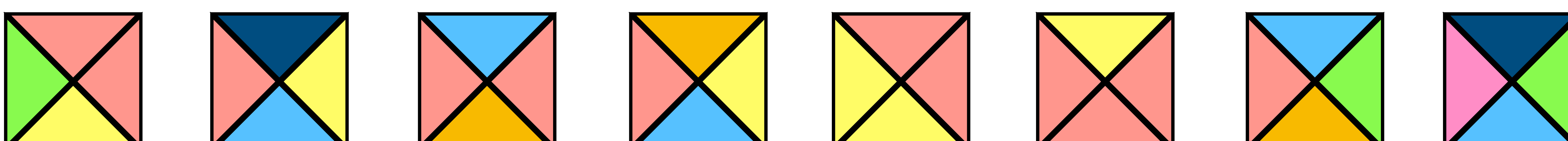
Given a finite set of tiles T, a tiling problem is the problem to determine whether T can tile the plane

Given a special tile τ^*, a recurring tiling problem is the problem to determine whether T can tile the plane such that τ^* appears infinitely often in the first column

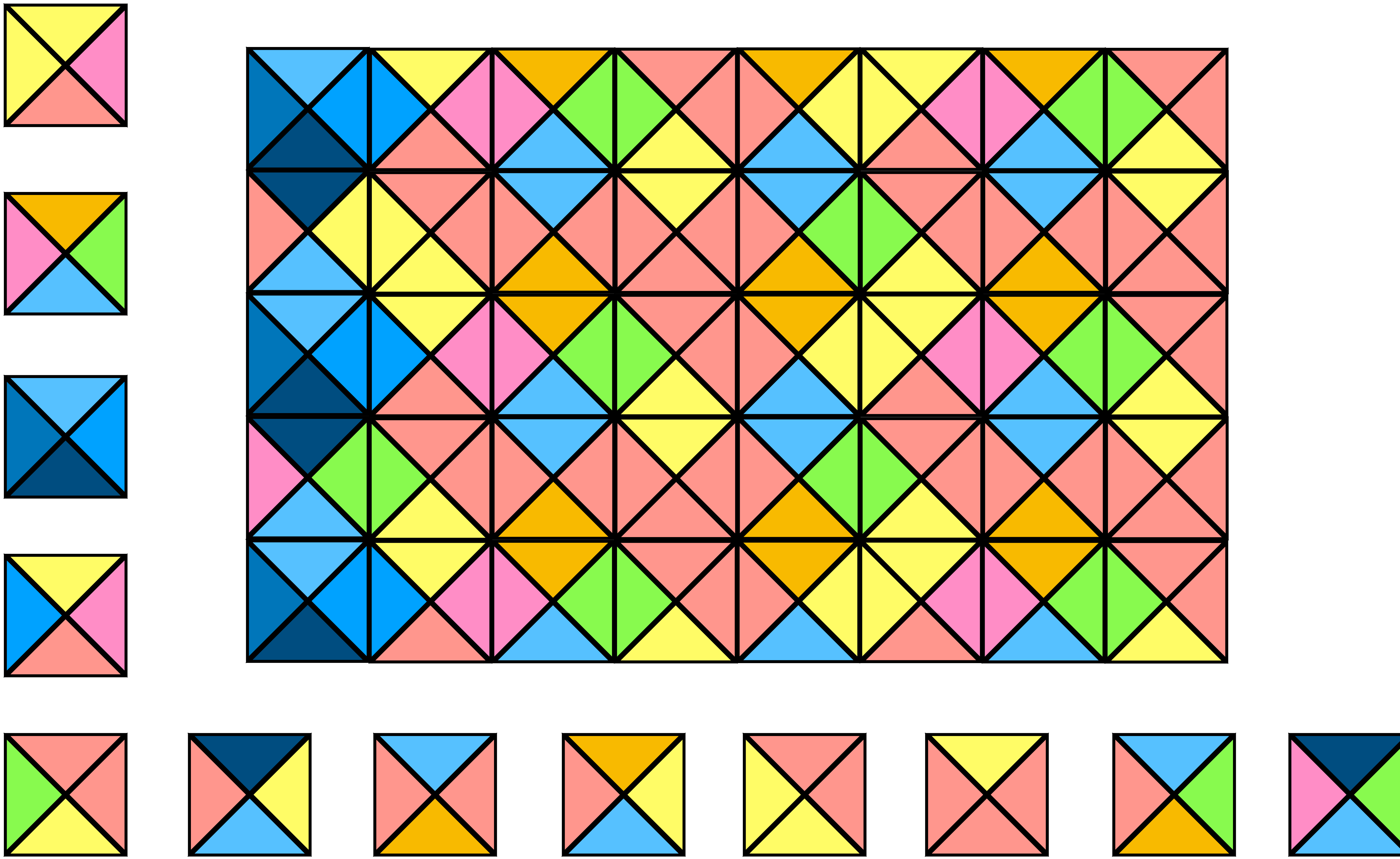
Recurring Tiling Problem



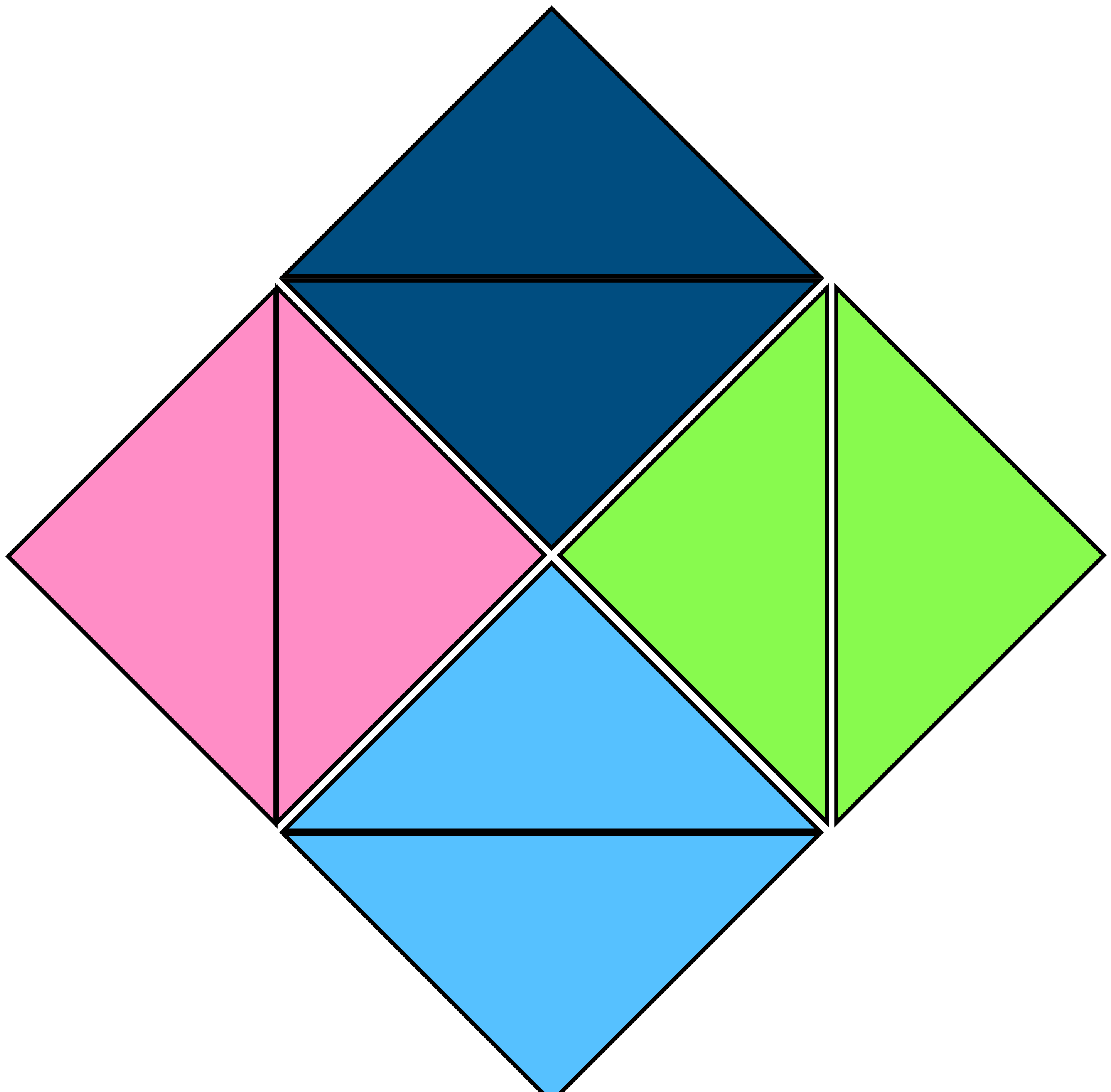
Can these tiles tile the plane such that  appears infinitely often in the first column?



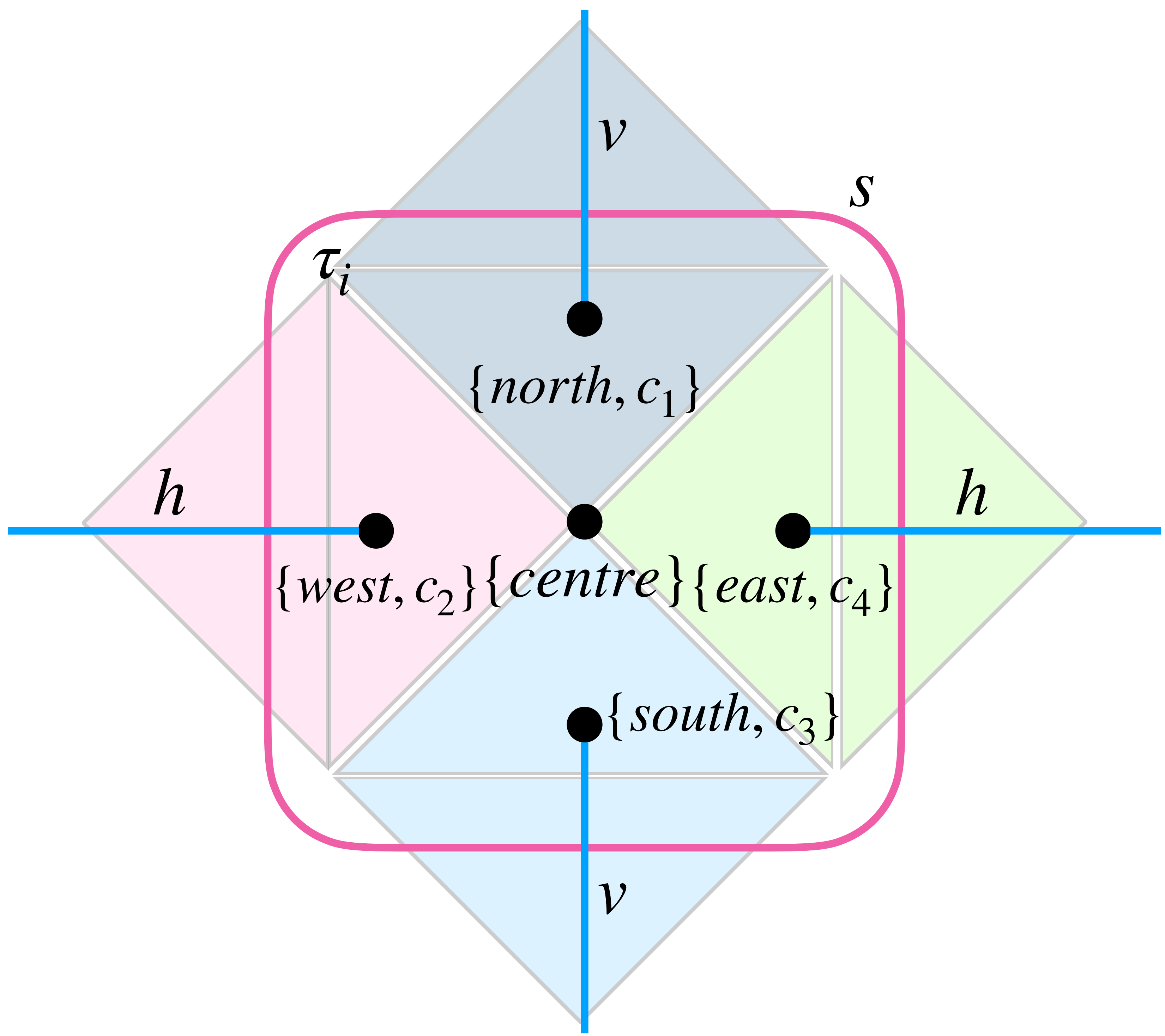
Recurring Tiling Problem



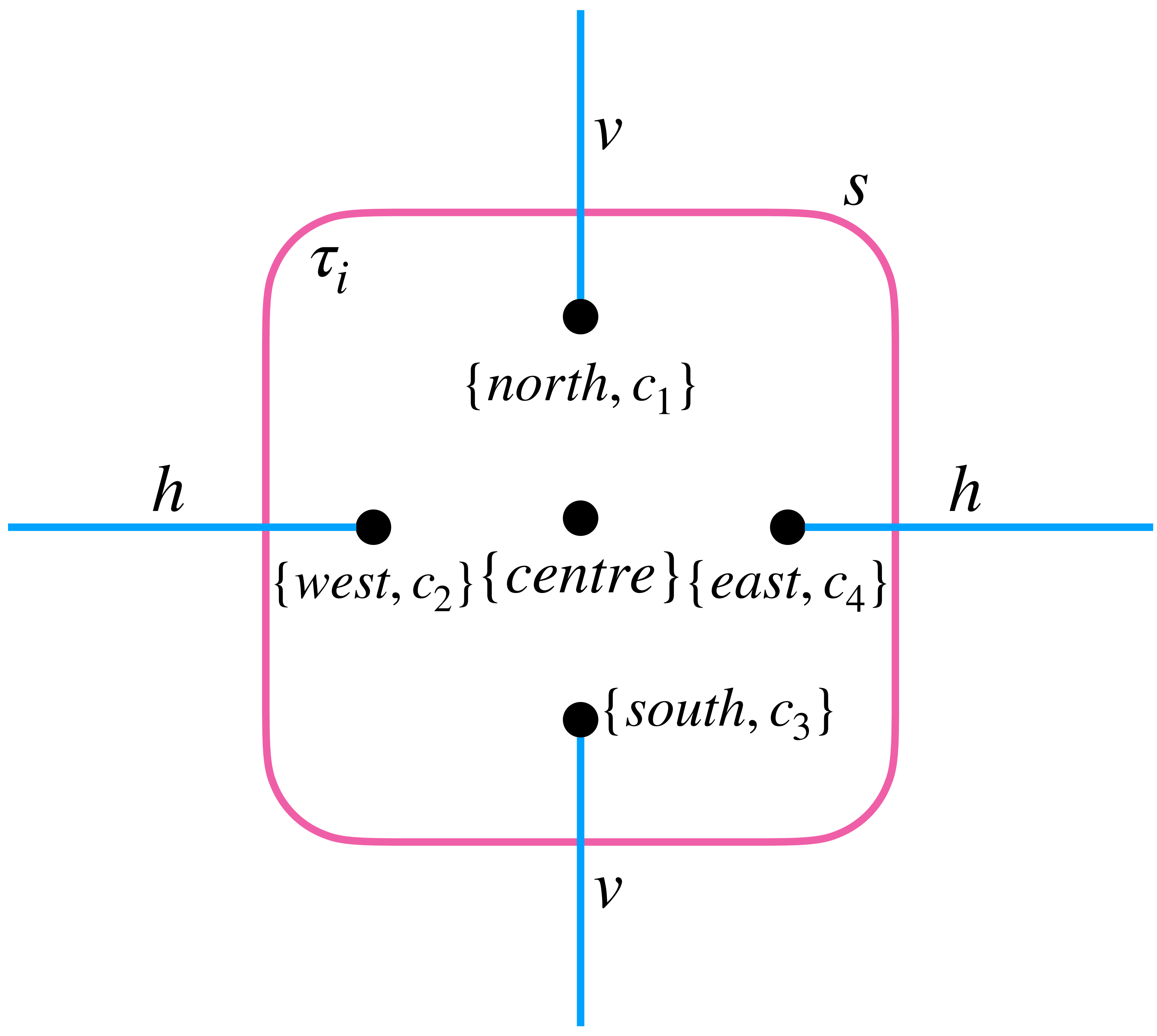
Encoding a Tiling



Encoding a Tiling



Encoding a Tiling



Encoding a Tiling

ψ_{tile} encodes the representation of a single tile

adj_tiles requires that adjoining tiles agree on colour

$init$ forces the existence of a tile at position $(0,0)$

$\psi_{x\&y}$ guarantees that making a move does not lead to different tiles

$tile\_left$ forces the special tile to appear only in the leftmost column

$$right\,\&\,up := [!](\Diamond_{right}\Diamond_{up}\texttt{centre} \rightarrow \Box_{up}\Box_{right}\,\texttt{centre})$$

Encoding a Tiling

ψ_{tile} encodes the representation of a single tile

adj_tiles requires that adjoining tiles agree on colour

$init$ forces the existence of a tile at position $(0,0)$

$\psi_{x\&y}$ guarantees that making a move does not lead to different tiles

$tile\_left$ forces the special tile to appear only in the leftmost column

$$\Psi_T := C_{\{h,v,s\}}(\psi_{tile} \wedge adj\_tiles \wedge init \wedge \psi_{x\&y} \wedge tile\_left)$$

Encoding a Tiling

$$\Psi_T := C_{\{h,v,s\}}(\psi_{tile} \wedge adj\_tiles \wedge init \wedge \psi_{x\&y} \wedge tile\_left)$$

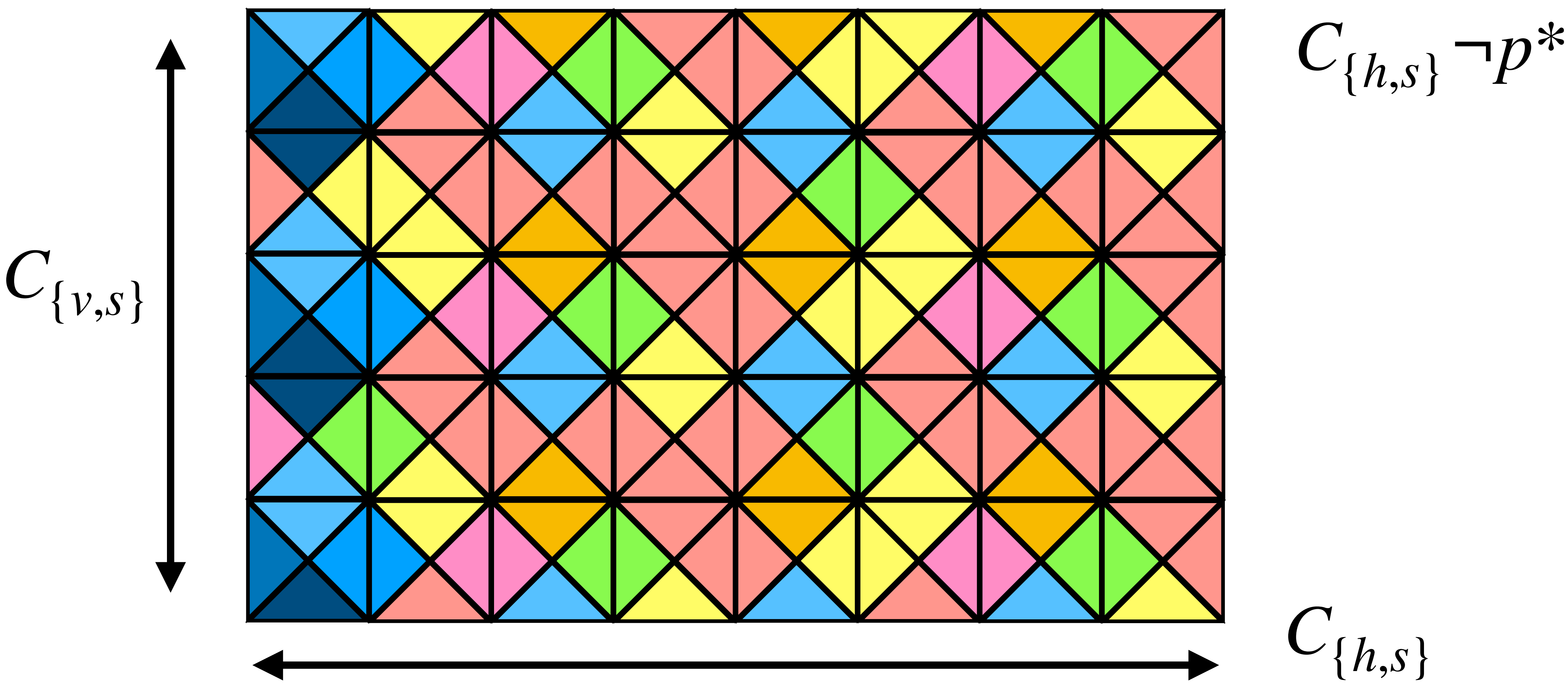
Lemma. If T can tile $\mathbb{N} \times \mathbb{N}$, then Ψ_T is satisfiable

Lemma. If Ψ_T is satisfiable, then T can tile $\mathbb{N} \times \mathbb{N}$

Encoding the Recurring Tile

$$\Psi_T \wedge C_{\{v,s\}}[C_{\{h,s\}} \neg p^*] \neg \Psi_T$$

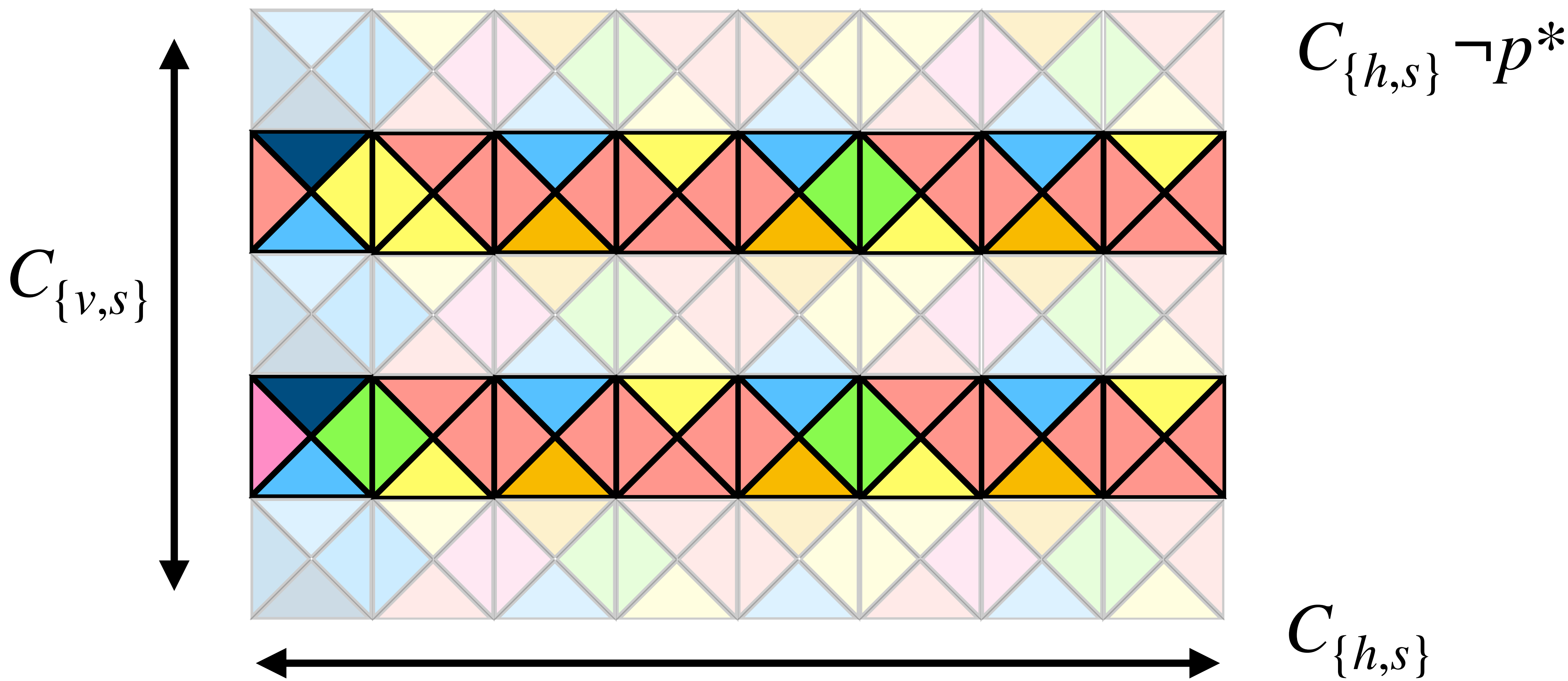
T can tile $\mathbb{N} \times \mathbb{N}$ and after removing all rows with the special tile (p^*) we no longer have a tiling



Encoding the Recurring Tile

$$\Psi_T \wedge C_{\{v,s\}}[C_{\{h,s\}} \neg p^*] \neg \Psi_T$$

T can tile $\mathbb{N} \times \mathbb{N}$ and after removing all rows with the special tile (p^*) we no longer have a tiling



Encoding the Recurring Tile

$$\Psi_T \wedge C_{\{v,s\}}[C_{\{h,s\}} \neg p^*] \neg \Psi_T$$

T can tile $\mathbb{N} \times \mathbb{N}$ and after removing all rows with the special tile (p^*) we no longer have a tiling

Theorem. T can tile $\mathbb{N} \times \mathbb{N}$ with τ^* appearing infinitely often in the first column if and only if $\Psi_T \wedge C_{\{v,s\}}[C_{\{h,s\}} \neg p^*] \neg \Psi_T$ is satisfiable

Theorem. Satisfiability of APALC is Σ^1_1-hard

Harel. *Effective transformations on infinite trees, with applications to high undecidability, dominoes, and fairness*, 1986.

Corollaries

Theorem. Satisfiability of APALC is Σ^1_1-hard

Corollary. The set of valid formulas of APALC is neither RE nor co-RE

Open Problem*. Is there a finitary axiomatisation of APAL with common knowledge? **NO!**

Corollary. GALC and CALC do not have finitary axiomatisations

RG and LBK. *Satisfiability of APAL with Common Knowledge is Σ^1_1-hard*, 2023.

Take-home message

- Adding common knowledge to APAL is not trivial
- However, common knowledge can be treated in an infinitary fashion
- Which fragment we quantify over, EL=PAL, ELC, or PALC, matters; increase in expressivity is not linear
- APALC is not finitely axiomatisable

Open Problem. Expressivity of APALC^{XX}