Quantification in Dynamic Epistemic Logic
Day 1

Rustam Galimullin & Louwe B. Kuijer

ESSLLI 2023

Hi.

This course

- Welcome to the “Quantification in Dynamic Epistemic Logic” course.
- I am Louwe Kuijer.
- I will be teaching this course together with Rustam Galimullin.

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Course overview

- 5 days, 1 lecture each.

  Day 1: Non-quantified DEL.

  Day 2: APAL and friends.

  Day 3: GAL and CAL.

  Day 4: Group knowledge.

  Day 5: AAML and AAUML.

- See course website for more details.
  (Linked from Discord and ESSLLI course catalogue.)

Further reading

- Most of this course is based directly on research papers (as opposed to textbooks and handbooks).
- As a result: not a lot of easy reading on this topic.
- Website does provide list of papers for further reading.
- But: expect those to be highly detailed and technical.

Exercises

- We have written some exercises that you can do to test yourself.
- They are, of course, completely optional.
- Solutions will not be published or discussed during the lectures.
- If you want to discuss the exercises: talk to us before or after the lecture.

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Epistemic logic

- Our starting point: epistemic logic (EL).
- Used to represent the information state of one or more agents *at a specific point in time*.

Epistemic logic: language

Definition

The *language of epistemic logic* (EL) is given by

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \Box_a \varphi$$

where $a \in A$ and $p \in P$.

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- As usual: $\wedge, \rightarrow, \leftrightarrow$ as abbreviations. Also: $\Diamond$ as dual of $\Box$.
- $\Box_a \varphi$ read as "agent a knows that φ (is true)".
- $\Diamond_a \varphi$ read as "agent a considers it possible that φ (is true)".

Epistemic logic: models

Definition

A *model* of epistemic logic is a triple $M = (S, \{R_a\}_{a \in A}, V)$ where

- S is a set of states (also called worlds),
- for each $a \in A$, $R_a \subseteq S \times S$ is an accessibility relations and
- $V : P \to 2^S$ is a valuation function.

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Note: in general, no reflexivity/transitivity/symmetry assumptions on R_a.
When we do assume that the relation is an equivalence, write $\sim_a$ for R_a.

Epistemic logic: semantics

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Definition

The *satisfaction relation* $\models$ is given by

$$
\begin{array}{lcl}
M, s \models p & \Leftrightarrow & s \in V(p), \\
M, s \models \neg\varphi & \Leftrightarrow & M, s \not\models \varphi, \\
M, s \models \varphi \vee \psi & \Leftrightarrow & M, s \models \varphi \text{ or } M, s \models \psi, \\
M, s \models \Box_a\varphi & \Leftrightarrow & \forall s' \in S\text{: if } (s, s') \in R_a \text{ then } M, s' \models \varphi.
\end{array}
$$

Example: cards (simple)

Situation:

- Agents: Rustam (r) and Louwe (l).
- Two cards from standard deck of playing cards, placed face down on table.
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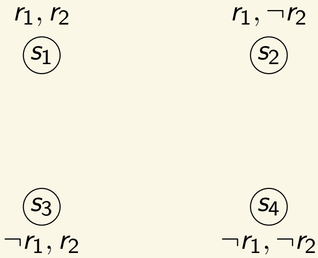

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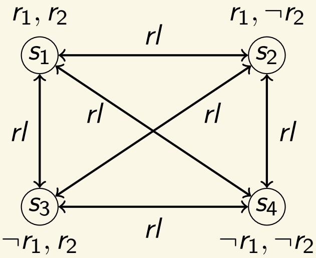

Information change

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- Model represents specific information state.
- Information change therefore requires model change.

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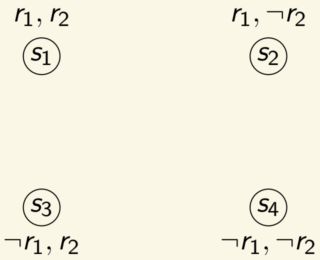

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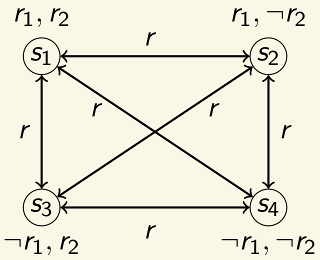

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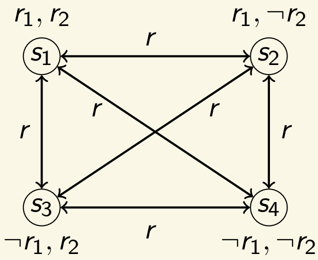

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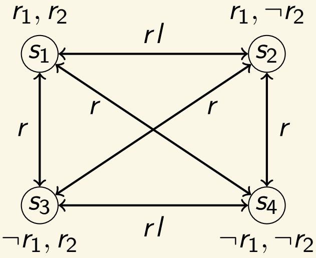

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Getting to the point

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- Answer: while they are simple, there is a point to them.
- Note that we *can* reason about information change using EL as opposed to DEL. (We just did.)
- But: it's relatively hard.

Reasoning about information change: EL vs. DEL

EL	
Ad-hoc	
Analyze twice	
Lots of effort	
Meta-logical	

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Model transformers

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- Initial model M_s turns into model $M * e_s$.
- In other words: e is a function that transforms models.

Updates as functions

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- $M = (S, R, V)$.
- Option 1: event changes S: public announcement.
- Option 2: event changes R_a: arrow updates.
- Option 3: event changes V: substitutions.
- Option 4: all of the above: action models, arrow update models. (Discussed later this week.)

Simplifying vs. "complexifying"

- Public announcements, arrow updates and substitution reduce, or at least do not increase, the complexity of a model.
- Action models and arrow update models do increase complexity.
- We start by considering the three simplifying update types.

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Public announcements

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- Specifically: announcement ψ restricts S to $S \cap [\![\psi]\!]_M$.

Public announcements: formally

Definition

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- $V * \psi(p) = V(p) \cap S * \psi$.

Public announcements: simple example

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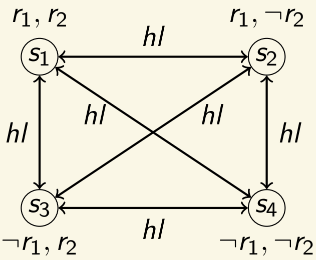

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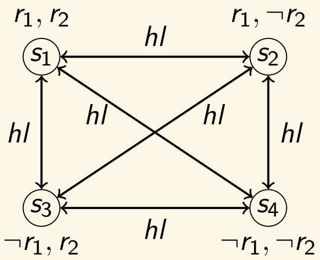

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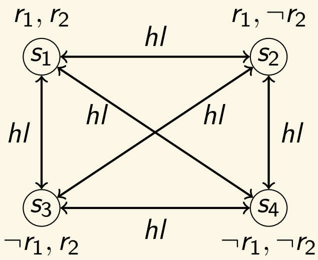

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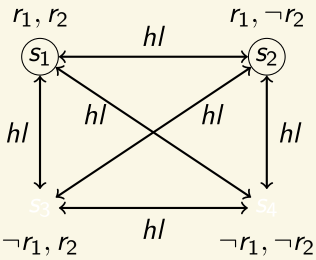

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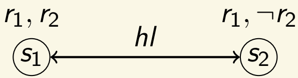

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Ad-hoc	Systematic
Analyze twice	Analyze two things
Lots of effort	Easy(ish)
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- But eventually we do of course want to add announcements to the language.

Public Announcement Logic

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The *language of public announcement logic* (PAL) is given by

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \Box_a\varphi \mid [\varphi]\varphi$$

where $a \in A$ and $p \in P$.

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- $\langle\varphi\rangle$ as dual of $[\varphi]$.

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$$\text{Equivalent to:} \quad M, s \models [\varphi]\psi \quad \Leftrightarrow \quad \text{if } M, s \models \varphi \text{ then } M * [\varphi], s \models \psi.$$

Ways to do information change

EL	Not yet PAL	PAL
Ad-hoc	Systematic	Systematic
Analyze twice	Analyze two things	Analyze two things
Lots of effort	Easy(ish)	Easy(ish)
Meta-logical	Meta-logical	In object language

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Next up: substitutions

- We have discussed public announcements.
- Arrow updates are more complicated, so we leave them for later.
- First, we discuss substitutions (a.k.a. assignments).

Factual change

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Factual change

- Public announcements change S.[1]
- Arrow updates change R.
- Substitutions change V.
- This means that substitutions represent *factual change* instead of *information change*.
- This course is about information change, so we won't say much about substitutions.
- But we do briefly discuss them for the sake of completeness.

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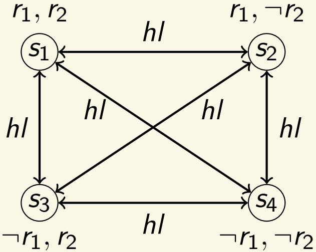
- Effect is global, i.e., common knowledge.

Substitutions: example 1



Substitutions: example 1

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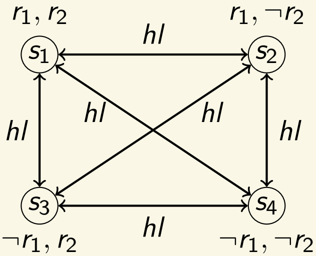

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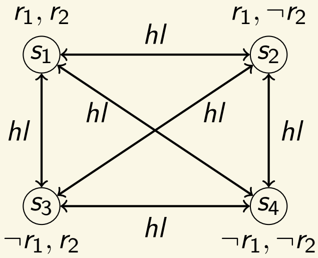

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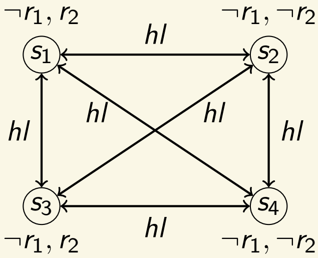
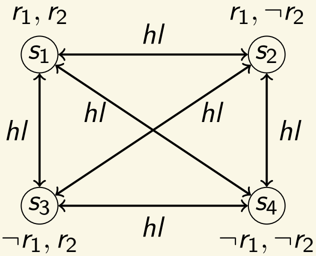


Substitutions: example 2



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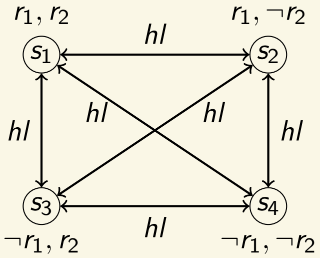

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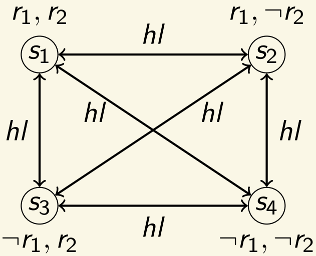

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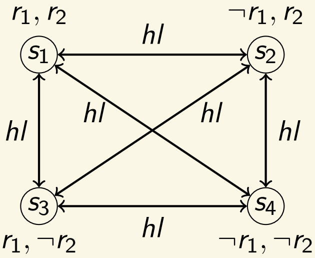

Substitutions in a logical language

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The *language of epistemic logic with factual change* (EL+$[\sigma]$) is given by

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where $[\sigma](M, s) = M * [\sigma], s$.

Systematic

- Note: as with PAL, we do not need substitutions in the language to do factual change systematically.
- But having them in the language still helps, by allowing in-logic reasoning.

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And now, arrow updates

- Finally, we arrive at arrow updates.

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- Hence this is an arrow update.

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- With public announcements, we specify what remains possible ($[\varphi]$ means φ worlds remain).
- We follow that convention for arrow updates.

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- Arrow update consists of set of such clauses.
- Every arrow matching no clause is deleted.

Arrow updates: formally

Definition

The language of arrow update logic (AUL) is given by

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \Box_a \varphi \mid [U]\varphi$$
$$U ::= \epsilon \mid U, \varphi \overset{a}{\mapsto} \psi$$

where $a \in A$, $p \in P$ and ϵ is the empty sequence.

Arrow updates: semantics

- $M * [U] = (W, R * [U], V)$
- $(s_1, s_2) \in R * [U]_a$ iff $(s_1, s_2) \in R_a$ and
  $\exists(\varphi \overset{a}{\mapsto} \psi) \in U : M, s_1 \models \varphi$ and $M, s_2 \models \psi$.

Satisfaction relation $\models$ is extended with

- $M, s \models [U]\varphi$ iff $M * [U], s \models \varphi$.

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- Similarly: $\neg r_1 \overset{l}{\mapsto} \neg r_1$.
- No further clauses: update U given by $U = \{\top \overset{r}{\mapsto} \top, r_1 \overset{l}{\mapsto} r_1, \neg r_1 \overset{l}{\mapsto} \neg r_1\}$.

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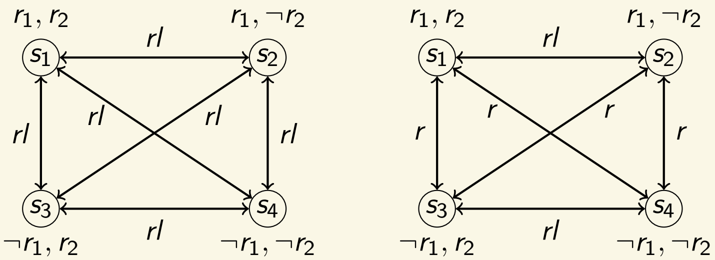
- We just established that $U = \{\top \overset{r}{\mapsto} \top, r_1 \overset{l}{\mapsto} r_1, \neg r_1 \overset{l}{\mapsto} \neg r_1\}$.



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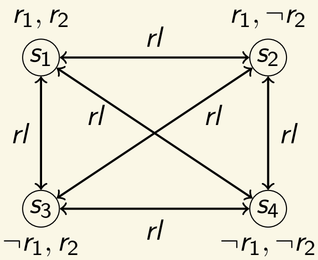

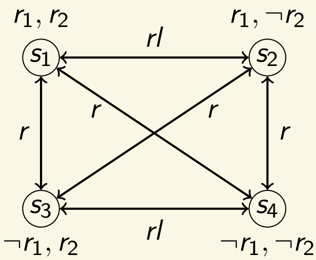

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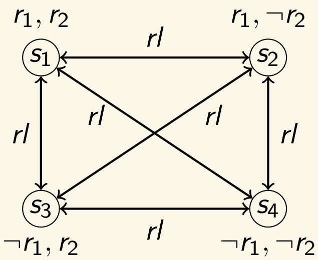

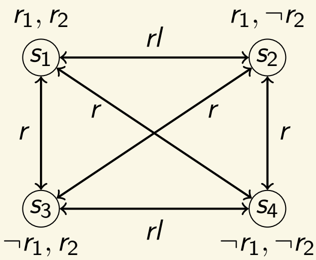

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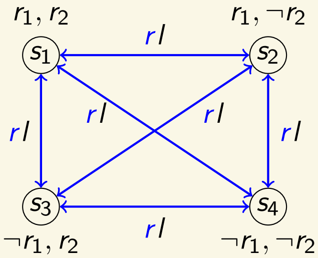

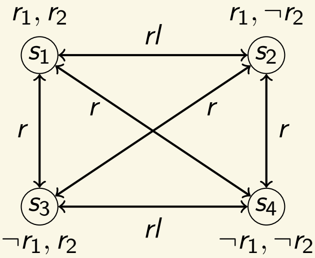

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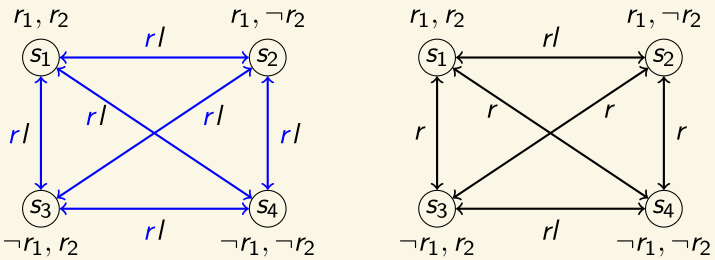

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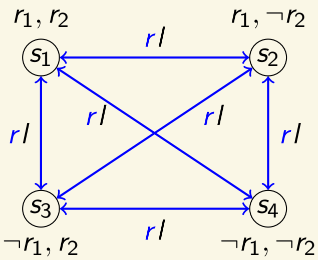

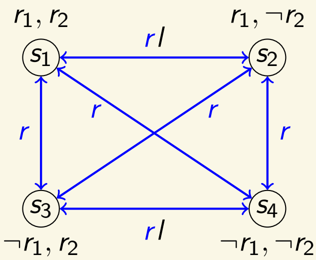

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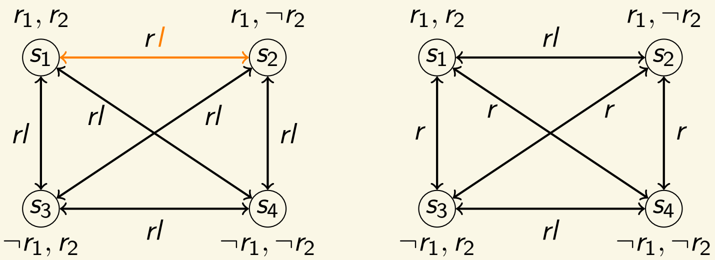

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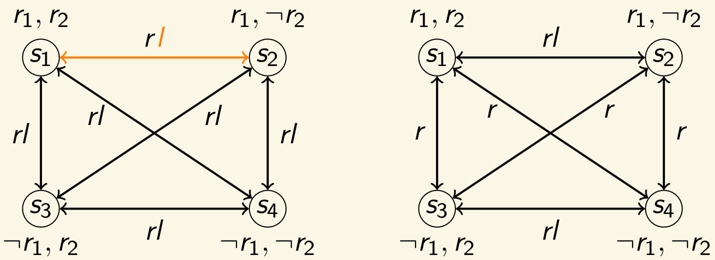

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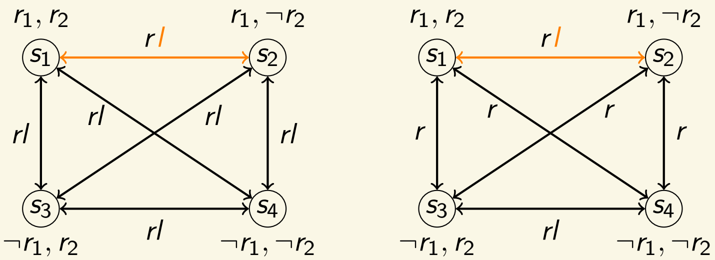

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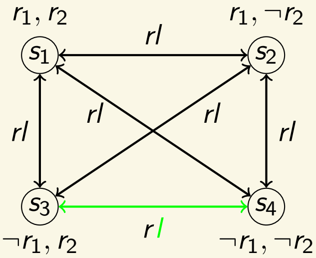

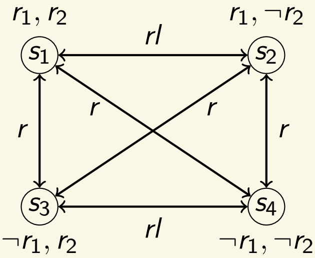

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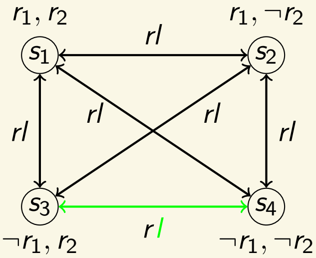

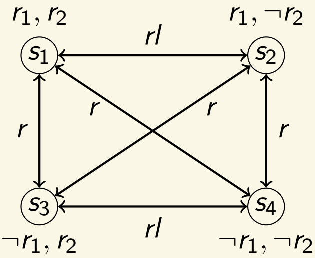

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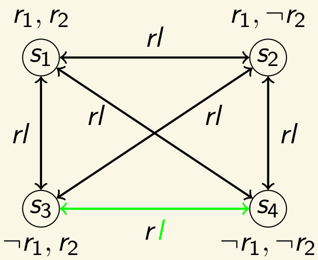

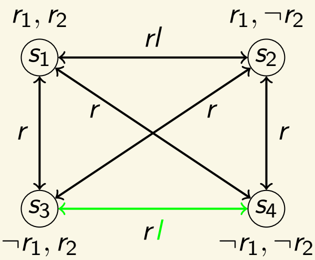

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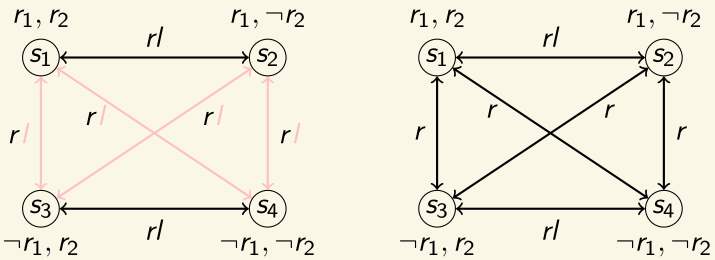

Table of Contents

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- Well known proof system **K**:

(Prop)	Any substitution instance of a validity of propositional logic
(K)	$\Box(\mathit{phi} \to \psi) \to (\Box\varphi \to \Box\psi)$
(Necc)	From $\vdash \varphi$, infer $\vdash \Box\varphi$
(MP)	From $\varphi \to \psi$ and φ, infer ψ

Completeness

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- Each line in proof is justified by (1) being a premise, (2) an axiom of **K** or (3) applying a rule of **K** to earlier line(s).

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- So, in some sense, all there is to know about basic modal logic.

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- These predictions can be encoded as axioms.

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- This implies that $\models [\sigma]\Box_a\varphi \leftrightarrow \Box_a[\sigma]\varphi$.

Axioms for Substitutions (IV)

Putting it all together:

$[\sigma]p \leftrightarrow \varphi$	where $p := \varphi$ in σ
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- Important property: in each axiom right-hand side has less complex formula inside scope of $[\sigma]$.
- Consequence: every formula with $[\sigma]$ is provably equivalent to one without.

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3. "Free" decidability: satisfiability of EL+$[\sigma]$ reduces to satisfiability of EL.

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- *Update* expressivity is about which model transformers can be expressed.
- Given a function $f : \mathfrak{M} \to \mathfrak{M}$, is there an update e in the language such that $[\![e]\!] = f$?

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Problem 1: equality too strong.

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Problem 2: public announcements are partial functions, not functions.

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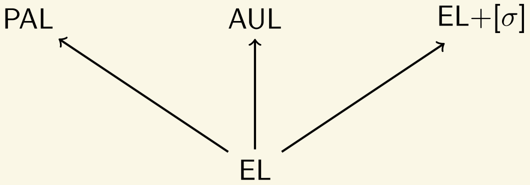

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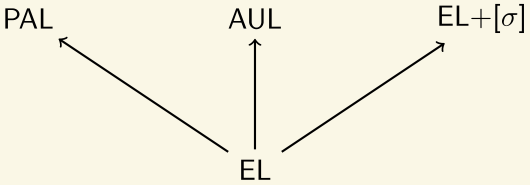

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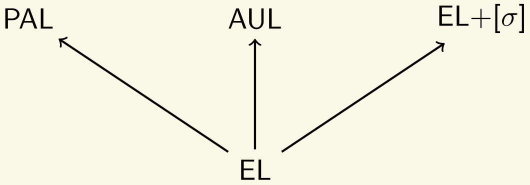

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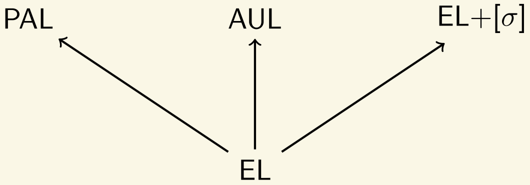

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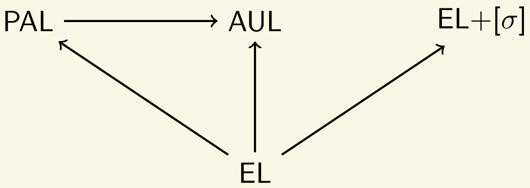

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- But: the four logics have different update expressivity.