Quantified Announcements and Common Knowledge

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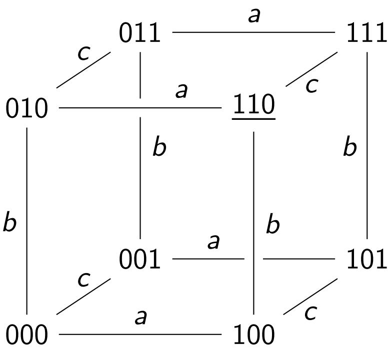
EL: Muddy Children

Muddy Children

Three children — Alice (a), Bobby (b), and Clair (c) — have been playing outside, and some of them may have mud on their foreheads. Each child can see every other child's forehead, but not their own.



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$M_{\underline{110}} \models \Box_a(m_b \wedge \neg m_c)$
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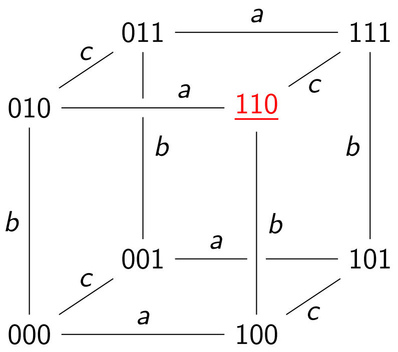

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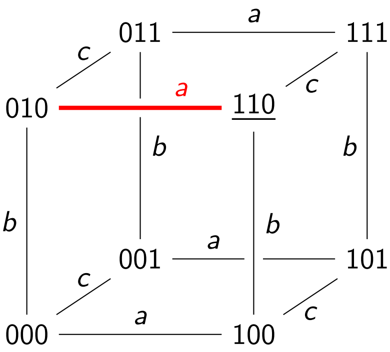

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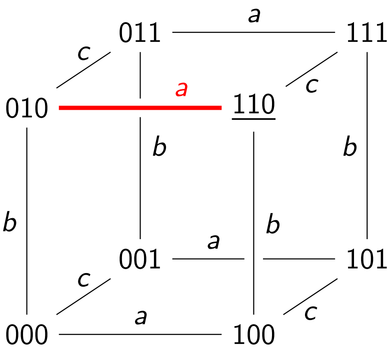

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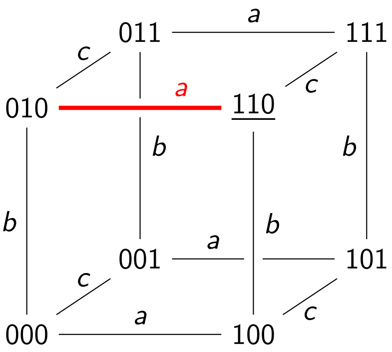

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EL: Syntax and Semantics

$$\mathcal{EL} \ni \varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_a\varphi$$

Definition (Semantics)

An epistemic model is a triple $M = (S, \sim, V)$, where

- S is a non-empty set of states,
- $\sim_a\colon A \to 2^{S\times S}$ is an equivalence relation for each agent a,
- $V : P \to 2^S$ is the valuation function.

A pair M_s with $s \in S$ is called a pointed model.

$$
\begin{array}{lll}
M_s \models p & \text{iff} & s \in V(p) \\
M_s \models \neg\varphi & \text{iff} & M_s \not\models \varphi \\
M_s \models \varphi \wedge \psi & \text{iff} & M_s \models \varphi \text{ and } M_s \models \psi \\
M_s \models \Box_a\varphi & \text{iff} & \forall t \in S : s \sim_a t \text{ implies } M_t \models \varphi
\end{array}
$$

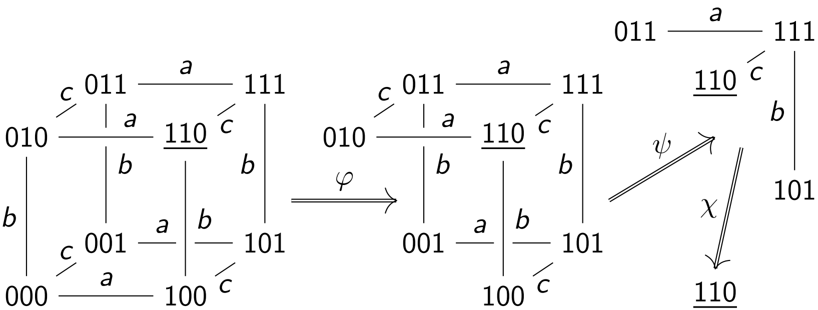
PAL: Muddy Children

Muddy Children Continued

Three children — Alice (a), Bobby (b), and Clair (c) — have been playing outside, and some of them may have mud on their foreheads. Each child can see every other child's forehead, but not their own... Then, their parent says: **'At least one of you has mud on their forehead'**, and, after that, **'If you know that you have a muddy forehead, please step forward'**. If nobody steps forward, the parent keeps repeating the request.

PAL: Muddy Children

'At least one child is muddy': $\varphi := m_a \vee m_b \vee m_c$
'Nobody steps forward': $\psi := \bigwedge_{i \in \{a,b,c\}} \neg(\Box_i m_i \vee \Box_i \neg m_i)$
'Alice and Bobby step forward': $\chi := \Box_a m_a \wedge \Box_b m_b$



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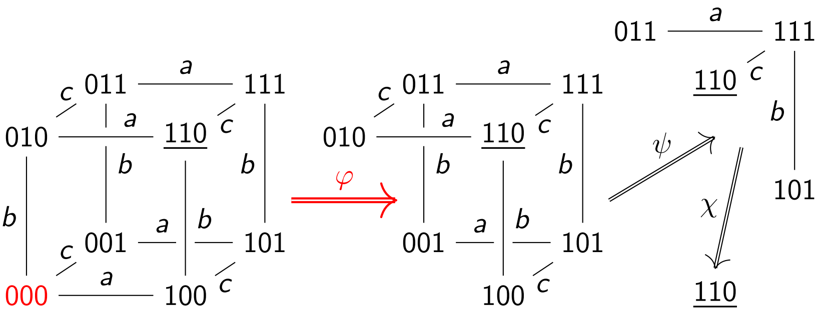

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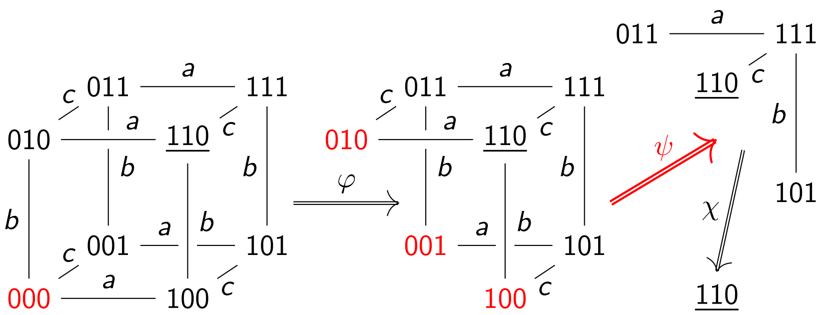

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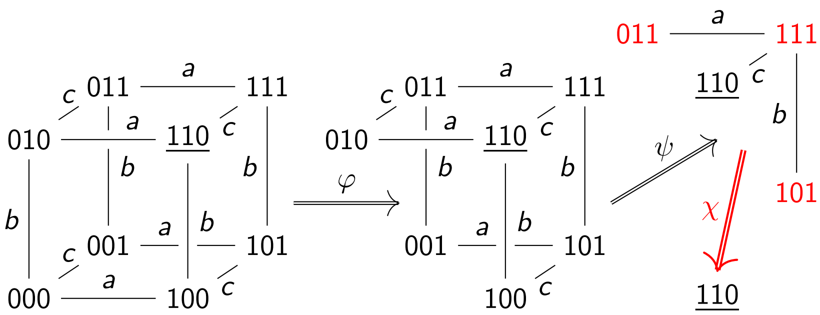


PAL: Syntax and Semantics

$$\mathcal{PAL} \ni \varphi ::= p \mid \neg\varphi \mid \varphi \land \varphi \mid \Box_a\varphi \mid [\varphi]\varphi$$

Definition (Semantics)

An announcement of φ in a pointed model M_s results in an updated pointed model M_s^φ containing only φ-states:

- $S^\varphi = [\![\varphi]\!]_M$,
- $\sim_a^\varphi = \sim_a \cap (S^\varphi \times S^\varphi)$,
- $V^\varphi(p) = V(p) \cap S^\varphi$.

$$M_s \models [\varphi]\psi \quad \text{iff} \quad M_s \models \varphi \text{ implies } M_s^\varphi \models \psi$$
$$M_s \models \langle\varphi\rangle\psi \quad \text{iff} \quad M_s \models \varphi \text{ and } M_s^\varphi \models \psi$$

PAL: Properties

For each $\varphi \in \mathcal{PAL}$ there is an equivalent $t(\psi) \in \mathcal{EL}$, where t is a translation function.

Theorem

PAL is sound and complete.

Theorem

PAL and EL are equally expressive.

GAL: Muddy Children

Strategic Muddy Children

Alice, Bobby, and Clair may not need a parent to learn who is muddy. **Can they communicate so that all of them know exactly who is muddy?** Of course. E.g. Clair can say that Alice and Bobby are muddy, and Alice can say that Clair's forehead is not muddy.
More generally, having an initial model M_s is there an announcement (or a sequence thereof) by a group (or groups) of agents such that some goal model N_s is reachable?

$$M_s \xRightarrow{\psi_1} M_s^{\psi_1} \xRightarrow{\psi_2} \dots \xRightarrow{\psi_n} N_s$$

GAL: Syntax and Semantics

Group Announcement Logic (GAL) = PAL + $\{[G]\varphi, \langle G\rangle\varphi\}$

$\langle G\rangle\varphi$: 'agents from G have a joint announcement such that φ holds in the resulting model'

$[G]\varphi$: 'whatever agents from G announce, they cannot avoid φ'

Let $\psi_G := \bigwedge_{a\in G} \Box_a \psi_a$, where ψ_a is an epistemic formula (truthfulness).

Definition (Semantics)

$$M_s \models [G]\varphi \quad \text{iff} \quad \forall \psi_G : M_s \models [\psi_G]\varphi$$
$$M_s \models \langle G\rangle\varphi \quad \text{iff} \quad \exists \psi_G : M_s \models \langle \psi_G\rangle\varphi$$

GAL: Properties

PAL	$[G]\varphi \to [\psi_G]\varphi$
From $\forall \psi_G : \eta([\psi_G]\varphi)$ infer $\eta([G]\varphi)$	From φ infer $[G]\varphi$

The axiomatisation is infinitary

Theorem

GAL is sound and complete.

Properties

$\langle G\rangle\langle G\rangle\varphi \leftrightarrow \langle G\rangle\varphi$ A multi-step strategy is reducible to a single-step strategy.
$\langle G \cup H\rangle\varphi \not\to \langle G\rangle\langle H\rangle\varphi$ It is not the case for different groups
$\langle G\rangle[H]\varphi \not\to [H]\langle G\rangle\varphi$ Confluence does not hold

The Notion(s) of Strategy

What does it mean that 'agents have a strategy'?

Nothing special: agents in a group guess their announcements.

Agents know that they have an announcement to achieve their goal (expressible in GAL).

If agents pool their knowledge together, they would know that they can achieve their goal.

It is their common (background) knowledge that they can achieve their goal.

Epistemic planning: what are the prerequisites for an execution of a plan?

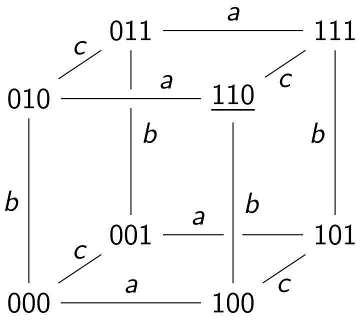
Common Knowledge

It is common knowledge that φ iff everyone knows φ, everyone knows that everyone knows φ, and so on.

The same thing a bit more formally

$\blacksquare_G \varphi := \bigwedge_n \square^n_G \varphi$ is not expressible in GAL (infinite formula).

$$M_s \models \blacksquare_G \varphi \quad \text{iff} \quad \forall n \in \mathbb{N} : M_s \models \square^n_G \varphi$$



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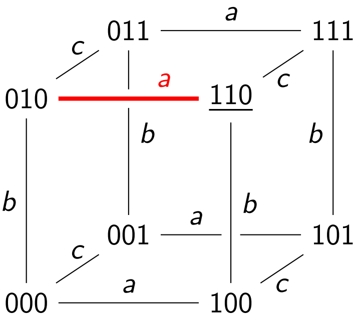

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$\blacksquare_G \varphi := \bigwedge_n \square^n_G \varphi$ is not expressible in GAL (infinite formula). We consider the transitive closure instead.

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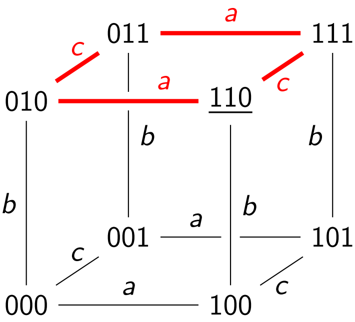

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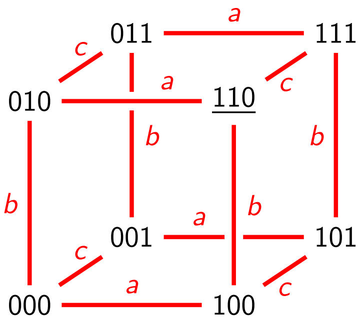

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Group Knowledge + Group Communication

GAL with Common Knowledge (GALC) = GAL + $\blacksquare_G \varphi$

Let $\psi_G := \bigwedge_{a \in G} \Box_a \psi_a$, where ψ_a is an epistemic formula.

Definition (Semantics)

$$M_s \models \blacksquare_G \varphi \quad \text{iff} \quad \forall n \in \mathbb{N} : M_s \models \Box^n_G \varphi$$
$$M_s \models [G]\varphi \quad \text{iff} \quad \forall \psi_G : M_s \models [\psi_G]\varphi$$

However, there is a subtlety...

Group Knowledge + Group Communication

GAL with Common Knowledge (GALC) = GAL + $\blacksquare_G \varphi$
Let $\psi_G := \bigwedge_{a \in G} \Box_a \psi_a$, where ψ_a is an **epistemic** formula.

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However, there is a subtlety... With the extended base language, we can also allow agents to announce formulas with $\blacksquare_G$ (still no circularity).

Group Knowledge + Group Communication

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Let $\psi_G := \bigwedge_{a\in G} \Box_a\psi_a$, where ψ_a is an **epistemic** formula.

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GALC eXtended (GALC^X) = PAL + $\{\blacksquare_G\varphi, [G]^X\varphi, \langle G\rangle^X\varphi\}$

Let $\psi_G := \bigwedge_{a\in G} \Box_a\psi_a$, where ψ_a is an **EL+C** formula.

Definition (Semantics)

$$M_s \models [G]^X\varphi \quad \text{iff} \quad \forall \psi_G : M_s \models [\psi_G]\varphi$$

GALC: Properties

General Case

$\langle G \rangle \varphi \rightarrow \langle G \rangle^X \varphi$ If a goal is achieved by simple means, it can be achieved by more complex means

$\Box_G \varphi \not\rightarrow \langle G \rangle \blacksquare_H \varphi$ Group knowledge cannot be made common...

$\Box_G \varphi \not\rightarrow \langle G \rangle \blacksquare_G \varphi$ ...even within the same group

$\blacksquare_G \varphi \not\rightarrow \langle G \rangle \blacksquare_H \varphi$ One cannot freely share common knowledge

Not all knowledge can be shared. The (in)famous offender $p \wedge \neg \Box_a p$ makes knowledge 'unstable'. What if we restrict our attention to 'stable' knowledge?

GALC: Properties

Positive (Universal) Fragment of ELC = box modalities + only propositions are negated
We cannot express that someone does not know something.

All φ's are positive

$\Box_G \varphi \to \langle G \rangle \blacksquare_H \varphi$
$\Box_G \varphi \to \langle G \rangle \blacksquare_G \varphi$
$\blacksquare_G \varphi \to \langle G \rangle \blacksquare_H \varphi$

GALC: Axiomatisation

PAL	$[G]\varphi \to [\psi_G]\varphi$
From $\forall \psi_G : \eta([\psi_G]\varphi)$ infer $\eta([G]\varphi)$	From φ infer $[G]\varphi$
From $\forall n \in \mathbb{N} : \eta(\Box_G^n \varphi)$ infer $\eta(\blacksquare_G \varphi)$	$\blacksquare_G \varphi \to \Box_G^n \varphi$

The axiomatisation is infinitary (as GAL). We do not rely on fixed-point axioms for common knowledge

Theorem

GALC is sound and complete.

GALC: Expressivity

Are GALC and GALC^X really different? Yes!

There are some properties expressible by GALC^X and not expressible by GALC. Vice versa: open question.

The Intuition

Formulas of logics are finite.
Quantification in the logics is implicit $\Rightarrow$ it ranges over an infinite number of propositions and arbitrary long formulas.

GALC: Overview

- GAL allows to reason about the existence of a public announcement that reaches certain epistemic goals
- GAL was extended with a classic group knowledge modality $\blacksquare_G \varphi$ (GALC), which means 'it is common knowledge among agents in group G that φ'
- Adding $\blacksquare_G \varphi$ leads to two possible interpretations of the semantics of group announcements: classical and extended (GALC^X)
- Axiomatisations of both GALC and GALC^X are sound and complete
- GALC and GALC^X are strictly more expressive than GAL. And GALC is not at least as expressive as GALC^X
- Similar results for APALC and APALC^X